The Berkowitz algorithm

September 30, 2021 4:41 PM

$$C(\lambda) = \det(A-\lambda I) \overset{T_1}{=} \det(A_r - \lambda I_r)\cdot(A_{nn}-\lambda) - R^T \operatorname{adj}(A_r - \lambda I_r)\, S.$$

$$A - \lambda I = \begin{bmatrix} \boxed{A_r - \lambda I_r} & S \\ -R^T- & A_{nn}-\lambda \end{bmatrix} \qquad r=n-1$$

$$\overset{T_2}{=} C_r(\lambda)(A_{nn}-\lambda) - R^T\left(-\sum_{k=1}^{r}\sum_{j=0}^{r-k} C_{r,k+j} A_r^{j}\lambda^{k-1}\right) S$$

$$= C_r(\lambda)(A_{nn}-\lambda) + \sum_{k=1}^{r}\sum_{j=0}^{r-k} C_{k+j}\,(R^T A_r^j S)\cdot\lambda^{k-1}.$$

compute recursively. $\qquad Q_j \in R.$

$Q_0 = R^T\cdot I_r\cdot S$
$Q_1 = R^T\cdot A_r\cdot S$
$Q_2 = R^T A_r^2\cdot S$
$Q_3 = R^T A_r^3\cdot S$

method ①

$B \leftarrow I_r \quad Q_0 \leftarrow R^T\cdot S$ —— r mults
for j to $r-1$ do
  $B \leftarrow A_r\cdot B$ // $B = A_r^j$ —— r^3 mults
  $Q_j \leftarrow R^T\cdot(B\cdot S)$ —— r^2+r mults.
od;

Costs $\displaystyle\sum_{r=1}^{n-1} r + (r-1)(r^3+r^2+r) \in O(n^5).$

$Q_3 \quad R^T(A_r\cdot(A_r\cdot(A_r\cdot S)))$
$\qquad Q_3 \quad Q_2 \quad Q_1$

Method ②

$Q_0 \leftarrow R^T\cdot S$ —— r mults
for j to $r-1$ do
  $S \leftarrow A_r\cdot S$ —— r^2 mults
  $Q_j \leftarrow R^T\cdot S$ —— r mults
od;

Cost $\displaystyle\sum_{r=1}^{n-1} r + (r-1)\cdot(r^2+r) \in O(n^4).$

Algorithm Berkowitz.

Algorithm Berkowitz.

Input $A \in R^{n \times n}$, R a ring.

Output $\det(A - \lambda I)$.

if $n = 1$ then output $A_{11} - \lambda$.

Let $A = \begin{bmatrix} A_r & S \\ -R^T- & A_{nn} \end{bmatrix}$, $r = n-1$.

Compute $C_r(\lambda) = \text{Berkowitz}(A_r) = \det(A_r - \lambda I_r)$.

$Q_0 = R^T \cdot S$

for j from 1 to $r-1$ do

  $S \leftarrow A_r \cdot S$

  $Q_j \leftarrow R^T \cdot S$

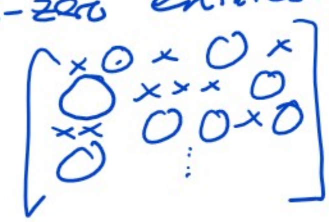
od;

$$C_n(\lambda) \leftarrow C_r(\lambda) \cdot (A_{nn} - \lambda) + \sum_{k=1}^{r} \left[\sum_{j=0}^{r-k} \overset{R.}{\overset{\Downarrow}{c_{r,k+j}}} \cdot Q_j \right] \lambda^{k-1}$$

output $C_n(\lambda)$.

Remark. If A is a <u>sparse matrix</u>

e.g. each row of A has 3 non-zero entries.



$Q_0 \leftarrow R^T \cdot S$ ——— r mults

for j from 1 to $r-1$ do

  $S \leftarrow A_r \cdot S$ ——— $3r$ mults

  $Q_j \leftarrow R^T \cdot S$ ——— r mults

od;

Cost: $\sum_{r=1}^{n} r + (r-1)(4r) \in O(n^3)$.