The adjugate matrix and characteristic polynomial

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The adjugate (adjoint) matrix adj(A).

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^{-1} = \frac{1}{ad-bc} \cdot \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\text{adj}(A)} = \frac{1}{\det(A)} \cdot \text{adj}(A).$

Def. Let $\overline{A}_{ij}$ denote the $(n-1)\times(n-1)$ submatrix of A obtained by deleting row i and column j.

Def. Let $C = \text{cof}(A)$ where $C_{ij} = (-1)^{i+j} \cdot \det(\overline{A}_{ij})$.

Eg. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$C_{11} = (-1)^{1+1} \cdot \det([d]) = +d.$

$C_{22} = (-1)^{2+2} \cdot \det([a]) = +a$

$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$C_{12} = (-1)^{1+2} \cdot \det([c]) = -c$

$C_{21} = (-1)^{2+1} \cdot \det([b]) = -b.$

Def. $\text{adj}(A) = C^T$.

Properties of adj(A).

(1) $\text{adj}(A) = \det(A) \cdot A^{-1} \Rightarrow A^{-1} = \text{adj}(A)/\det(A).$

(2) $\text{adj}(AB) = \text{adj}(B)\,\text{adj}(A).$

Proof (2). $\text{adj}(AB) = \det(AB) \cdot (AB)^{-1} = \det(A) \cdot \det(B)\, B^{-1} A^{-1}$

$= (\det(B) \cdot B^{-1})(\det(A)\, A^{-1})$

$= \text{adj}(B) \cdot \text{adj}(A)$

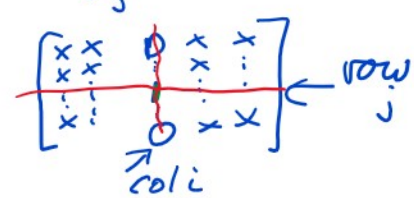
$\updownarrow$

(1). One way to compute A^{-1} is to solve

$$A \cdot X = I$$

$$\Rightarrow A \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix} \Rightarrow A x_j = e_j \Rightarrow A \underbrace{\begin{bmatrix} x \\ x \\ x \\ x \\ x \end{bmatrix}}_{x_j} = \begin{array}{l} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow j \end{array}$$

$$\overset{CR}{\Rightarrow} x_{ji} = \det\Big(\big[A_1 \cdots \underset{\text{col } i}{e_j} \cdots A_n\big]\Big)/\det(A).$$



$\det(A) = \det(A^T)$

$$= (-1)^{i+j} \cdot 1 \cdot \det(\overline{A}_{ji})/\det(A)$$

$$= (-1)^{i+j} \cdot \det(\overline{A}_{ij})/\det(A)$$

$$= C_{ij}/\det(A).$$

$= C_{ij} / \det(A).$

$\Rightarrow \quad A^{-1} = X = \mathrm{adj}(A)/\det(A).$

Theorem 1. Let $A \in R^{n\times n}$, R a ring. Let $A = \begin{bmatrix} \boxed{A_r} & \boxed{S} \\ \boxed{-R^T-} & \boxed{A_{nn}} \end{bmatrix}$ $\quad A_r = \overline{A}_{nn}$, $r = n-1$.
Then

$$\det(A) = \det(A_r)\cdot A_{nn} - \det(R^T \mathrm{adj}(A_r)\cdot S).$$

Ex. $\begin{array}{ccc} & A_r & S \\ & \begin{bmatrix} \boxed{a} & \boxed{b} \\ \boxed{c} & d \end{bmatrix} & \\ & R^T & \end{array}$ $\quad \det(A) = a\cdot d - \det(\boxed{c}\cdot \mathrm{adj}([a])\cdot \boxed{b})$
$\quad = ad - cb.$ $\qquad \mathrm{adj}([a]) = [1]$

$A = [a] \quad C_{11} = (-1)^{1+1}\cdot \det(\cdot) = 1$
$\mathrm{cof}(A) = \underbrace{[1]}_{C_{11}} \Rightarrow \mathrm{adj}(A) = C^T = [1]$

Proof Exercise.

Characteristic Polynomials

$$C(\lambda) = \det(A - \lambda I) = \sum_{i=0}^{n} c_i \lambda^i.$$

The eigenvalues of A are the n roots of $C(\lambda)$.
The Berkowitz algorithm will compute $C(\lambda)$ using $O(n^4)$ ring operations $+, -, \times$.

Now $\left.\begin{array}{l} C(0) = \det(A - 0I) = \det(A). \\ C(0) = c_0 \end{array}\right] \Rightarrow \det(A) = c_0$

E.g. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det(A - \lambda I) = \det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right)$

$= (a-\lambda)(d-\lambda) - bc = ad - a\lambda - d\lambda + \lambda^2 - bc$
$= \underbrace{1}_{c_2}\lambda^2 \underbrace{-(a+d)}_{c_1}\lambda + (\underbrace{ad-bc}_{c_0})\cdot\lambda^0$
$\qquad ad - bc = \det(A).$

Theorem 2. Let $A = \begin{bmatrix} A_r & \boxed{S} \\ \boxed{R^T} & \boxed{A_{nn}} \end{bmatrix}$ where $A_r = \overline{A}_{nn}$

Remark 2. Let $A = \begin{bmatrix} A' & b \\ \boxed{R^T} & \boxed{A_{nn}} \end{bmatrix}$ where $A_r = A_{nn}$

$$\mathrm{adj}(A_r - \lambda I) = -\sum_{k=1}^{r} \sum_{j=0}^{r-k} C_{k+j} A_r^{j} \lambda^{k-1} \quad \text{where } C(\lambda) = \det(A_r - \lambda I_r)$$

Example. $A_r = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $r = 2$.

$$\mathrm{adj}\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = \begin{bmatrix} d-\lambda & -b \\ -c & a-\lambda \end{bmatrix}.$$

$$\mathrm{adj}\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = \underbrace{-C_1 \cdot A_r^0 \cdot \lambda^0}_{k=1,\ j=0} \underbrace{- C_2 A_r^1 \lambda^0}_{k=1,\ j=1} \underbrace{- C_2 A_r^0 \lambda^1}_{k=2,\ j=0}$$

matrix

$$= (a+d)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot 1 - 1 \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot 1 - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \lambda$$

$$= \begin{bmatrix} a+d-a-\lambda & 0-b-0 \\ 0-c-0 & a+d-d-\lambda \end{bmatrix} = \begin{bmatrix} d-\lambda & -b \\ -c & a-\lambda \end{bmatrix}$$