Fraction Free GE Algorithms

September 22, 2021 10:01 PM

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    |\^/|     Maple 2021 (X86 64 LINUX)
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 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
> n := 5;
                                  n := 5

> A := Matrix(n,n,rand(10^4));
                       [7926    8057       5    3002    2347]
                       [                                    ]
                       [9765    3354    5860    6906    5281]
                       [                                    ]
                  A := [5393    1203     311    9386    9810]
                       [                                    ]
                       [5144    7995    3121    9390    2055]
                       [                                    ]
                       [6505    5293    2987    2440    8012]

> det := LinearAlgebra[Determinant](A);
                          det := 23791466233143137296

> mu := 1:
> for k to n-1 do
>     mu := mu*A[k,k]^(n-k-1);
>     for i from k+1 to n do
>         for j from k+1 to n do
>             A[i,j] := A[k,k]*A[i,j]-A[i,k]*A[k,j];
>         od;
>         A[i,k] := 0;
>     od;
>     print(A[k+1..n,k+1..n]);
>     printf("max length = %d digits\n",length(max(seq(seq(abs(A[i,j]),j=k+1..n),i=k+1..n))));
> od:
                [-52092801    46397535    25422426    18938751]
                [                                             ]
                [-33916423     2438021    58203650    65096689]
                [                                             ]
                [ 21923162    24711326    58982852     4214962]
                [                                             ]
                [-10458467    23642437     -188570    48235877]

max length = 8 digits
         [ 1446635080430484    -2169753403021452    -2748734175828216]
         [                                                          ]
         [-2304462863969796    -3629921935279464     -634766482939224]
         [                                                          ]
         [ -746353677117192      275702742865512    -2314671639266760]

max length = 16 digits
     [-10251288552034411837957632844368 , -72526312932221072093434374683 52]
     [                                                                     ]
     [-12205621711824463499652169349 76 ,  -54000130525879127873828042013 12]

max length = 32 digits
        [4650480458878993983728364631799031577175100679050779659553 1264]

max length = 62 digits
> det = A[n,n]/mu;
                  23791466233143137296 = 23791466233143137296
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Let $A^{(0)} = A$ and $A^{(k)}$ be the matrix after the k'th step

Ordinary Gaussian elimination:

$$R_i \leftarrow R_i - \frac{A_{ik}^{(k-1)}}{A_{kk}^{(k-1)}} \cdot R_k$$

for $k = 1$ to $n-1$ (step k)
  for $i = k+1$ to n (row i)
    for $j = k+1$ to n
      $A_{ij}^{(k)} := A_{ij}^{(k-1)} - \frac{A_{ik}^{(k-1)}}{A_{kk}^{(k-1)}} A_{kj}^{(k-1)}$ ← pivot

$$\begin{array}{cc} & \begin{array}{cccc} & k & & j \end{array} \\ \begin{array}{c} \\ k \\ \\ i \end{array} & \begin{bmatrix} 4 & x & & x \\ 0 & \boxed{3} & & 2 \\ 0 & x & & x \\ 0 & \boxed{5} & & 11 \end{bmatrix} \end{array}$$

$$= 11 - \frac{5}{3} \cdot 2 = \frac{23}{3}$$

Clear fractions

$$R_i \leftarrow A_{kk}^{(k-1)} R_i - A_{ik}^{(k-1)} R_j$$

$$A_{ij}^{(k)} := A_{kk}^{(k-1)} A_{ij}^{(k-1)} - A_{ik}^{(k-1)} A_{kj}^{(k-1)}$$

$$\det(A^{(k)}) = \left(A_{kk}^{(k-1)}\right)^{n-k} \det(A^{(k-1)})$$

Bareiss/Edmonds: $A_{00}^{(-1)} := 1$

$$A_{ij}^{(k)} := \frac{A_{kk}^{(k-1)} A_{ij}^{(k-1)} - A_{ik}^{(k-1)} A_{kj}^{(k-1)}}{A_{k-2\,k-2}^{(k-2)}}$$

current pivot ← previous pivot

Theorem (Jack Edmonds 1967, Erwin Bareiss 1968, Jordan 1838–1922)
The division by $A_{k-2\,k-2}^{(k-2)}$ is exact in R and $\det(A) = A_{nn}^{(n-1)}$

Moreover $A_{kk}^{(k-1)} = \det(k \times k \text{ principle minor of } A)$ and

$$\underset{i>k,\, j>k}{A_{ij}^{(k-1)}} = \det(k \times k \text{ minor of } A)$$

Example.

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 1 \\ 5 & 3 & 1 \end{bmatrix} \begin{array}{l} R_2 \leftarrow (2R_2 - 3R_1)/1 \\ R_3 \leftarrow (2R_3 - 5R_1)/1 \end{array} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -5 & -7 \\ 0 & 1 & -13 \end{bmatrix} R_3 \leftarrow (-5R_3 - 1R_2)/2 \begin{bmatrix} 2 & 1 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & 72/2 \end{bmatrix}$$

$= 36$

$$-7 = \det\left(\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}\right)$$

Proof that $A_{k-2,k-2}^{(k-2)}$ divides exactly for $n=3$.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{array}{l} R_2 \leftarrow aR_2 - dR_1 \\ R_3 \leftarrow aR_3 - gR_1 \end{array} \begin{bmatrix} a & b & c \\ 0 & \underset{U}{\overset{S}{ae-db}} & \underset{V}{\overset{T}{af-dc}} \\ 0 & ah-gb & ai-gc \end{bmatrix} \sim \begin{bmatrix} a & b & c \\ 0 & S & T \\ 0 & 0 & \boxed{} \end{bmatrix}$$

$$A_{33} \leftarrow (SR_3 - UR_2)/a$$

$u \qquad v$

$$A_{33} \leftarrow (sR_3 - uR_2)/a$$

$$= \left[(ae-db)(ai-gc) - (ah-gb)(af-dc)\right]/a$$

$$= \left(a^2ei - aegc - adbi + \cancel{bdgc} - ahaf + ahdc + agbf \cancel{- gbdc}\right]/a$$