Determinants

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Computational Linear Algebra.

Let A be an $n \times n$ matrix over a commutative ring R.
E.g. $R = \mathbb{Z}$ and $R = \mathbb{Z}[x_1, \ldots, x_n]$.

How fast can we

1. decide if $\det(A) = 0$?
2. compute $\det(A)$ and $\operatorname{rank}(A)$.
3. solve $Ax = b$
4. compute $\det(A - \lambda I)$.
5. multiply $A \cdot B$.

(1) The Schwarz – Zippel Lemma. (1978)

Let D be an integral domain and f be a non-zero polynomial in $D[x_1, \ldots, x_n]$. Let S be a finite subset of D. Let α be chosen at random from S^n.

$$\operatorname{Prob}[\, f(\alpha) = 0 \,] \le \frac{\deg(f)}{|S|}.$$

Suppose $A_{ij} \in \mathbb{Z}[x_1, \ldots, x_n]$

Suppose $d \geq \deg \det(A)$

Use $S = [0, 1000)$

If $\det(A) \neq 0$ then

$$\operatorname{Prob}[\, \det(A(\alpha)) = 0 \,] \le \frac{d}{|S|} = \frac{4}{1000} = \boxed{\frac{1}{250}}.$$

$$A = \begin{bmatrix} u & v^2 & w \\ v & u & s \\ w & s^2 & u \end{bmatrix}.$$

$d = 5$ $\quad d = 4.$

Repeat for k different $\alpha \in S^n$ chosen randomly.

(2) How fast can we compute $\det(A)$?

Def. (cofactor expansion)

$$A = \begin{array}{c} \\ - \\ + \\ \\ \end{array} \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}$$

Let $\bar{A}_{ij}$ = submatrix of A with row i and column j deleted.

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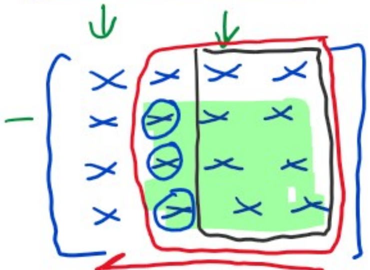
$$\det(A) = A_{11}\cdot\det(\bar{A}_{11}) - A_{21}\cdot\det(\bar{A}_{21}) + A_{31}\cdot\det(\bar{A}_{31}) - A_{41}\cdot\det(\bar{A}_{41})$$

Let $M(n)$ be the #mults in R that this alg. does.

[x] $\left.\begin{array}{l} M(1) = 0 \\ M(n) = n + n\,M(n-1) \end{array}\right\} \Rightarrow \sim 1.72 \cdot n! \in O(n!)$.

?

Gentleman & Johnson 1976



$M(1)=0$

$M(2)=2$

$$M(n) = \sum_{i=2}^{n} \boxed{i} \binom{n}{i} = \frac{n2^n}{2} - n \in O(n\cdot 2^n)$$

$i \times i$ submatrices

There are $\binom{4}{2}=6$ 2×2 determinants.

cofactor

$$M_{cf}(10)/M(10) = \frac{6,235,300}{5,110} = \underline{\underline{1,220}}$$

E&J.

Determinant(A, method=minor) ← Maple.

Bottom up implementation

1. Compute all $\binom{n}{2}$ 2×2 minors.
2. Compute all $\binom{n}{3}$ 3×3 minors.
   → Forget (garbage collect) all 2×2 results.
3. Compute all $\binom{n}{4}$ 4×4 minors.
   → Forget all 3×3 results.

⋮

Ordinary Gaussian Elimination over a field.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 1 \\ 5 & 3 & 1 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - \frac{5}{2}R_1]{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & \boxed{-5/2} & -\frac{7}{2} \\ 0 & 1/2 & -\frac{13}{2} \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + \frac{1}{5}R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -5/2 & -7/2 \\ 0 & 0 & -\frac{36}{5} \end{bmatrix}$$

$$A = \begin{bmatrix} \cdots \\ 3 & -1 & 1 \\ 5 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - \frac{5}{3}R_1} \begin{bmatrix} \cdots \\ 0 & \boxed{\begin{matrix} -1/2 & 2 \\ 1/2 & -\frac{13}{2} \end{matrix}} \end{bmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{bmatrix} \cdots \\ 0 & 0 & \frac{-36}{5} \end{bmatrix}$$

Th. $\det(A) = \det(B) = \det(C) = \cancel{2} \cdot \left(-\frac{5}{\cancel{2}}\right) \cdot \left(\frac{-36}{\cancel{5}}\right) = +\underline{36}.$

$$M(n) = (n-1)^2 + (n-2)^2 + \dots + 1 = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n \in O(n^3)$$

Can we compute $\mathrm{Det}(A)$ using $O(n^3)$ ring operations $+, -, \times$??

Don't know.

Berkowitz (1984) $O(n^4)$ operations.

Kaltofen (1992) $O(n^{3.5})$ operations.

The Bareiss/Edmonds algorithm for R an integral domain e.g. $\mathbb{Z}$.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ \underline{3} & -1 & 1 \\ 5 & 3 & 1 \end{bmatrix} \xrightarrow[R_3 \leftarrow 2R_3]{R_2 \leftarrow 2R_2} \underbrace{\overbrace{\begin{bmatrix} 2 & 1 & 3 \\ 6 & -2 & 2 \\ 10 & 6 & 2 \end{bmatrix}}^{R_2 \leftarrow 2R_2 - 3R_1}}_{R_3 \leftarrow 2R_3 - 5R_1} \xrightarrow[R_3 \leftarrow R_3 - 5R_1]{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -5 & -7 \\ 0 & 1 & -13 \end{bmatrix}$$

B　　　　C

$$\longrightarrow \det(B) = \det(C) = \boxed{2^2} \det(A)$$

$$C = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -5 & -7 \\ 0 & 1 & -13 \end{bmatrix} \xrightarrow{R_3 \leftarrow -5R_3} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -5 & -7 \\ 0 & -5 & +65 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & 72 \end{bmatrix}$$

D　　　　E

$$\det(E) = \det(D) = -5 \det(C)$$

$$\det(E) = 2 \cdot (-5) \cdot 72 \quad \Rightarrow \det(C) = \frac{\det(E)}{-5} = 2 \cdot 72.$$

$$\det(A) = \frac{\det(C)}{4} = \frac{2 \cdot 72}{4} = \underline{36}$$

We computed $\det(A)$ using twice as many $\times$ as Ord. G.E.

It does $O(n^3)$ mults and a few divisions.

But the size of the integers grows exponentially!!

See handout