Cost of Bareiss/Edmonds

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Cost. $M(n) = 2(n-1)^2 + M(n-1) \Rightarrow \frac{2}{3}n^3 - n^2 + \frac{1}{3}n \in O(n^3)$.
$M(1) = 0$

How big are the $A_{ij}^{(k)}$ if $A_{ij}^{(0)} \in \mathbb{Z}$?

<u>Hadamard's bound for $A_{ij} \in \mathbb{Z}$.</u>

Suppose $|A_{ij}| < B^m$ (m digits base B).

$$|\det(A)| \le \prod_{i=1}^{n} \sqrt{\sum_{j=1}^{n} A_{ij}^2}$$

Then $|\det(A)| \le \prod_{i=1}^{n} \sqrt{\sum_{j=1}^{n} B^{2m}} = \left(\sqrt{n B^{2m}}\right)^n = \sqrt{n}^{\,n} \cdot B^{mn}$

$\log_B |\det(A)| \le mn + \frac{n}{2} \log_B n^{<1} \le mn + \frac{n}{2} \in O(mn)$.

$\Rightarrow \log_B A_{ij}^{(k)} \sim km$ digits base B. $\sim mn$ digits base B.

$$\text{Cost}: \sum_{k=1}^{n-1} O(\underset{\substack{\uparrow \\ \text{size at step } k}}{mk})^2 \underset{\substack{\uparrow \\ \# \text{ops}}}{(n-k)} = O\left(\frac{m^2}{30}(n^5 - n)\right) \in O(m^2 n^5).$$

cost of $\times$ and $\div$ $\leftarrow$ 2

<u>The achilles heal of the Bareiss/Edmonds algorithm.</u>

$$A_{ij}^{(k)} = \frac{A_{kk}^{(k-1)} A_{ij}^{(k-1)} - A_{ik}^{(k-1)} \cdot A_{kj}^{(k-1)}}{A_{k-2,k-2}^{(k-2)}}$$

Step $k = n-1$: $i = n$, $j = n$

$$\underset{\substack{\uparrow \\ \det}}{A_{nn}^{(n-1)}} = \frac{A_{n-1,n-1}^{(n-2)} A_{nn}^{(n-2)} - A_{n,n-1}^{(n-2)} A_{(n-1,n)}^{(n-2)}}{A_{n-2\,n-2}^{(n-2)}} = D$$

$$D = \underset{\substack{\uparrow \\ \det(n-2 \times n-2 \text{ principle minor})}}{A_{n-2\,n-2}^{n-2}} \cdot \underset{\substack{\uparrow \\ \det(A)}}{A_{nn}^{n-1}} = \prod \text{ of two determinants.}$$

If $A = T_n = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_2 & x_1 & x_2 & & \vdots \\ x_3 & x_2 & x_1 & \ddots & x_2 \\ \vdots & & \ddots & & \end{bmatrix}$ an $n \times n$ symmetric Toeplitz matrix.

T_3

$\det(T_n) \in \mathbb{Z}[x_1, \ldots, x_n]$

$$T_3 \quad \begin{bmatrix} \ddots & & & & \\ x_3 & x_2 & x_1 & \ddots & x_2 \\ \vdots & & \ddots & & \\ x_n & \cdots & x_3 & x_2 & x_1 \end{bmatrix} \qquad \det(T_n) \in \mathbb{Z}[x_1, \ldots, x_n]$$

$$D = \det(T_n) \cdot \det(T_{n-2})$$

$n=8$: D has 19,175 terms; $\det(T_n)$ has 1520 terms; $\det(T_{n-2})$ has 120 terms.

Perhaps E&J algorithm is faster than B&E algorithm for det(T_n)?

To run the Bareiss/Edmonds algorithm in Maple use

Determinant(A, method=fracfree)