Monomial orderings

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for $R[x,y,z,\dots]$

$$f = 3x^4 + 7x^2y^1z^2 + 8x^2y^2z^1 + 2y^3z^3$$

$\deg_x$	4	2	2	0
$\deg$	4	5	5	6

plex $\quad 3x^4 + 8x^2y^2z^1 + 7x^2y^1z^2 + 2y^3z^3$

grlex $\quad 2y^3z^3 + 8x^2y^2z^1 + 7x^2y^1z^2 + 3x^4$

$x > y > z$

Define
$LC(f)$ = the leading coefficient of f.
$LM(f)$ = " " monomial of f
$LT(f)$ = " " term of f $= LC(f)\cdot LM(f)$.

	LM	LC	LT
$<_{plex}$	x^4	3	$3x^4$
$<_{grlex}$	y^3z^3	2	$2y^3z^3$.

Lemma 1. Let X, Y, Z be monomials. $<_{plex}$ and $<_{grlex}$ sat.

(i) $1 \le X$ (well ordering)

(ii) $X < Y \Rightarrow X\cdot Z < Y \cdot Z$

(iii) $<$ is a total ordering. (see Wiki).

Lemma 2. Let $a, b \in R[x_1, \dots, x_n]$. If R is an int. dom. then
$LC(a\cdot b) = LC(a)\cdot LC(b)$.
$LM(a\cdot b) = LM(a)\cdot LM(b)$.

Example of polynomial $\times$ and $\div$.

$x > y$ plex.

$$h = (\underbrace{x^3}_{f_1} + \underbrace{x^2y}_{f_2} + \underbrace{xy^2}_{f_3})(\underbrace{xy}_{g_1} + \underbrace{y^2}_{g_2} + \underbrace{y}_{g_3})$$

$f \qquad g$

$$f_1\cdot g = x^4\cdot y + x^3y^2 + x^3y$$

plex.

$$f_1 \cdot g = x^4 y + x^3 y^2 + x^3 y$$
$$+\ f_2 \cdot g = x^3 y^2 + x^2 y^3 + x^2 y^2 \qquad \text{MERGE}$$
$$x^4 y + 2x^3y^2 + x^3 y + \underline{x^2y^3} + \underline{x^2y^2}$$
$$+\ f_3 g = +\ \underline{x^2 y^3} + xy^4 + xy^3 \qquad \text{MERGE.}$$
$$= \underline{x^4y + 2x^3y^2 + x^3y + 2x^2y^3 + x^2y^2 + xy^4 + xy^3}.$$

$$\overset{=g}{xy + y^2 + y}$$
$$f = x^3 + x^2y + xy^2 \;\big)\; x^4y + 2x^3y^2 + x^3y + 2x^2y^3 + x^2y^2 + xy^4 + xy^3 = h$$
$$\underline{xy \cdot f} = -(x^4y + x^3y^2 + x^2y^3) \Leftarrow \text{MERGE}$$
$$0 + x^3y^2 + x^3y + \cancel{x^2y^3} + x^2y^2 + \cancel{xy^4} + xy^3$$
$$-(x^3y^2 + \cancel{x^2y^3} + \cancel{xy^4}) \qquad \text{MERGE}$$
$$0 + x^3y + x^2y^2 + xy^3$$
$$\underline{-(x^3y + x^2y^2 + xy^3)}$$
$$0.$$

$\Rightarrow f \mid h$ with quotient g and remainder 0.

If in the middle we had a remainder

$$f = x^3 + \cdots \;\big)\; \underbrace{x^2y^3 + \cdots}_{\bar{r}}$$

In the middle we've computed some terms of q say $\bar{q}$ and under the $\sqrt{}$ we have $\bar{r}$ and $h = \bar{q} \cdot f + \bar{r}$

$$h - \bar{q} f = \bar{r}$$

Claim: if $LM(f) \nmid LM(\bar{r})$ and $h = \bar{q} f + \bar{r}$ then $f \nmid \bar{r}$

Proof: TAC suppose $f \mid \bar{r} \Rightarrow \bar{r} = f \cdot \hat{q}$ for some $\hat{q}$.

$$\Rightarrow LM(\bar{r}) = LM(f \cdot \hat{q}) = LM(f) \cdot LM(\hat{q}) \quad \boxed{\times}$$

to remainder

$$f = \underline{xy} + \cdots \qquad \big)\; \bar{r} = x^3 + x^2 y$$