Monomial Orderings

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CLO 2.2 <u>Monomial Orderings</u>

Let M be a set of monomials in $k[x_1, \ldots, x_n]$, i.e.,

$$M = \{ x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} : \alpha \in \mathbb{Z}^n_{\geq 0} \}.$$

α is called an exponent vector.

Def An order relation $<$ on M is a <u>total ordering</u> if $\forall\ x^\alpha, x^\beta, x^\gamma \in M$

(1) Either $x^\alpha < x^\beta$ or $x^\alpha > x^\beta$ or $x^\alpha = x^\beta$

(2) $x^\alpha > x^\beta$ and $x^\beta > x^\gamma \Rightarrow x^\alpha > x^\gamma$.

Suppose $f = 2x^\alpha + 3x^\beta + 4x^\gamma$ where $\alpha > \beta > \gamma$.

Def. $LT(f) = 2x^\alpha$; $LC(f) = 2$, $LM(f) = x^\alpha$.

We'll define monomial orderings on $\mathbb{Z}^n_{\geq 0}$ rather than x^α.

Def 1. A <u>monomial ordering</u> on $k[x_1, \ldots, x_n]$ is a relation $>$ on $\mathbb{Z}^n_{\geq 0}$ s.t.

(i) $>$ is a total ordering $\Rightarrow LM(f)$ is unique

(ii) $\forall\ \alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$ $\alpha > \beta \Rightarrow \gamma + \alpha > \gamma + \beta$ $\Rightarrow LM(fg) = LM(f) \cdot LM(g)$.

(iii) Every non-empty subset $S \subset \mathbb{Z}^n_{\geq 0}$ has a least element under $>$. $\Rightarrow$ ($\div$ alg. terminates).
(a well ordering)

Ex. In $k[x]$ there is only one monomial ordering, namely

$$\cdots\cdots x^4 > x^3 > x^2 > x^1 > 1.$$

Notice $1 > x > x^2 > x^3 > \cdots$ satisfies (i), (ii) but not (iii).

In $k[x_1, \ldots, x_n]$ with $n \geq 2$ there are an infinite # of monomial orderings. Let $p_1, p_2, \ldots, p_n$ be n distinct primes.

Define $x^\alpha > x^\beta \Leftrightarrow \prod p_i^{\alpha_i} > \prod p_i^{\beta_i}$.

Notice the monomial $1 = x_1^0 x_2^0 \cdots x_n^0 \longrightarrow 1$.

Define $x^\alpha > x^\beta \iff \prod p_i^{\alpha_i} > \prod p_i^{\beta_i}$.

Notice the monomial $1 = x_1^0 x_2^0 \cdots x_n^0 \longrightarrow 1$.

$$\overset{8}{x^3} < \overset{9}{y^2} < \overset{10}{xz} \qquad\qquad \overset{27}{x^3} < \overset{25}{y^2} < \overset{21}{xz}$$

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5 \qquad\qquad p_1 = 3,\ p_2 = 5,\ p_3 = 7$$

Lemma 2. A relation $>$ on $\mathbb{Z}^n_{\geq 0}$ is a well ordering $\iff$ every strictly decreasing sequence

$$\alpha^{(1)} > \alpha^{(2)} > \alpha^{(3)} > \cdots\cdots \quad \text{is finite.}$$

Pf. We will prove $>$ is not a well ordering $\iff$ $\exists$ an infinite strictly decreasing sequence in $\mathbb{Z}^n_{\geq 0}$.

$(\Rightarrow)$ Given $>$ is not a well ordering

$\Rightarrow \exists\ S = \{\alpha^{(1)}, \ldots\} \subset \mathbb{Z}^n_{\geq 0}$ with no least elem.

$\Rightarrow \exists\ \alpha^{(2)} \in S$ s.t. $\alpha^{(1)} > \alpha^{(2)}$

$\Rightarrow \exists\ \alpha^{(3)} \in S$ s.t. $\alpha^{(2)} > \alpha^{(3)}$ etc.

$\Rightarrow \alpha^{(1)} > \alpha^{(2)} > \alpha^{(3)} > \cdots\cdots$ an infinite strictly dec. seq.

$(\Leftarrow)$ Given an inf. str. dec. seq.

$$\alpha^{(1)} > \alpha^{(2)} > \alpha^{(3)} > \cdots\cdots$$

Let $S = \{\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \cdots\cdots\}$. S is a set with no least element.

<u>Lexicographical Order.</u> Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ with $\alpha \neq \beta$.

Then $\alpha >_{lex} \beta$ if the left-most non-zero element in $\alpha - \beta$ is +ve and $\alpha <_{lex} \beta$ if " " " " " " " " is −ve.

E.g.

$$x^1 y^2 z^1 >_{lex} x^1 z^4$$

$$\alpha = [1,2,1] \quad [1,0,4] = \beta.$$

$$\alpha - \beta = [0,2,-3]$$

<u>Graded lex. order</u> Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ with $\alpha \neq \beta$.

Graded lex. order Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ with $\alpha \neq \beta$.

Def. $\deg(\alpha) = \sum \alpha_i = \deg(x^\alpha)$.

Then $\alpha >_{grlex} \beta$ if $\deg(\alpha) > \deg(\beta)$ or

$\deg(\alpha) = \deg(\beta)$ and $\alpha >_{lex} \beta$.

E.g. $xz^4 >_{grlex} xy^2z$

$\alpha = [1,0,4]$ $\quad$ $\beta = [1,2,1]$

$\deg \alpha = 5$ $\quad$ $\deg(\beta) = 4$

Prop 4. Lex. order is a monomial ordering.

Proof (iii) TAC suppose $>_{lex}$ is NOT a well ordering.

Then by Lemma 2 there is an infinite strictly decreasing sequence in $\mathbb{Z}^n_{\geq 0}$ i.e.

$$\alpha^{(1)} > \alpha^{(2)} > \alpha^{(3)} > \cdots\cdots$$

$$S = [\underline{\alpha_1^{(1)}}, \alpha_2^{(1)}, \ldots, \alpha_n^{(1)}] > [\underline{\alpha_1^{(2)}}, \alpha_2^{(2)}, \ldots, \alpha_n^{(2)}] > \cdots$$

In $>_{lex}$ $\alpha_1^{(1)} \geq \alpha_1^{(2)} \geq \alpha_1^{(3)} \geq \cdots$ where $\alpha_1^{(i)} \in \mathbb{Z}_{\geq 0}$.

But $\mathbb{Z}_{\geq 0}$ is a well ordering $\Rightarrow$ This sequence must "Stabilize" (stops decreasing), i.e., $\exists k \geq 1$ s.t.

$$\alpha_1^{(k)} = \alpha_1^{(k+1)} = \alpha_1^{(k+2)} = \cdots\cdots$$

Now consider S starting at k i.e.

$$\alpha^{(k)} > \alpha^{(k+1)} > \alpha^{(k+2)} > \cdots\cdots$$

$$[\underline{\alpha_1^{(k)}}, \underline{\alpha_2^{(k)}}, \ldots, \alpha_n^{(k)}] > [\underline{\alpha_1^{(k)}}, \underline{\alpha_2^{(k+1)}}, \ldots, \alpha_n^{(k+1)}] > [\underline{\alpha_1^{(k)}}, \underline{\alpha_2^{(k+2)}}, \ldots$$

In lex $\alpha_2^{(k)} \geq \alpha_2^{(k+1)} \geq \alpha_2^{(k+2)} \geq \cdots\cdots$

$\mathbb{Z}_{\geq 0}$ is a well ordering hence this sequence must stabilize, at index $\ell \geq k$. I.e.

$$\alpha_2^{(\ell)} = \alpha_2^{(\ell+1)} = \alpha_2^{(\ell+2)} = \cdots$$

Repeating this argument n times, the sequence S must stabilize i.e. stop decreasing, a contradiction.

Ex. Let $>$ be a relation on $\mathbb{Z}_{\geq 0}^n$ that satisfies props. (i) and (ii) in Def' (monomial ordering).

Show that $>$ is a well ordering $\Longleftrightarrow$ the monomial $[0, 0, \ldots, 0] \leq \alpha \quad \forall \alpha \in \mathbb{Z}_{\geq 0}^n$.