MATH 895, Assignment 3, Summer 2019

Instructor: Michael Monagan

Please hand in the assignment by 5:00pm Wednesday June 12th.
Late Penalty -20% off for up to 48 hours late, zero after than.
For Maple problems, please submit a printout of a Maple worksheet containing your Maple code and Maple output. Use any tools from the Maple library, e.g. `content(...)`, `Content(...) mod p`, `divide(...)`, `Divide(...) mod p`, `Eval(...) mod p`, `Interp(...) mod p`, `chrem(...)`, `Linsolve(A,b) mod p`, `Roots(f) mod p`, etc.

Question 1: Brown's dense modular GCD algorithm

REFERENCE: Section 7.4 of the Geddes text and Brown's original paper: On Euclid's algorithm and the computation of polynomial greatest common divisors. W. S. Brown, *Journal of the ACM* **18**(4), pp. 478–504, 1971. (see course webpage)

(a) (5 marks)

Let $a, b \in \mathbb{Z}[x]$, $g = \gcd(a, b)$, $\bar{a} = a/g$ and $\bar{b} = b/g$. For the modular GCD algorithm in $\mathbb{Z}[x]$ we said a prime p is *unlucky* if $\deg(\gcd(\bar{a} \bmod p, \bar{b} \bmod p))) > 0$ and a prime p is *bad* if $p|\mathrm{lc}(g)$. We apply Lemma 7.3 to identify the unlucky primes.

For $a, b, \in \mathbb{Z}[x_1, x_2, ..., x_n]$ we need to generalize these definitions for bad prime and unlucky prime and also define bad evaluation points and unlucky evaluation points for evaluating x_n. We do this using the lexicographical order monomial ordering. Let $g = \gcd(a, b)$, $a = \bar{a}g$ and $b = \bar{b}g$. Let's use an example in $\mathbb{Z}[x, y, z]$.

$$g = 5xz + yz - 1, \quad \bar{a} = 3x + 7(z^2 - 1)y + 1, \quad \bar{b} = 3x + 7(z^3 - 1)y + 1.$$

Let LC, LT, LM denote the leading coefficient, leading term and leading monomial respectively in lexicographical order with $x > y > z$. So in our example, $LT(a) = (5xz)(3x) = 15x^2z$, hence $LC(a) = 15$ and $LM(a) = x^2z$.

Let p be a prime. We say p is *bad prime* if p divides $LC(g)$ and p is an *unlucky prime* if $\deg(\gcd(\phi_p(\bar{a}), \phi_p(\bar{b})) > 0$ where deg here means total degree. Identify all bad primes and all unlucky primes for the example.

Suppose we have picked $p = 11$ and we evaluate at $z = \alpha \in \mathbb{Z}_{11}$. We think of a, b as elements of $\mathbb{Z}_p[z][x, y]$ with coefficients in $\mathbb{Z}_p[z]$. Define bad and unlucky evaluation points appropriately and identify them for example.

(b) (5 marks)

Prove the following modified Lemma 7.3 for $\mathbb{Z}[x_1, ..., x_n]$.

> Let $a, b \in \mathbb{Z}[x_1, ..., x_n]$ with $a \neq 0, b \neq 0$ and $g = \gcd(a, b)$. Let $LC(a)$ and $LM(a)$ denote the leading coefficient and leading monomial of a in lexicographical order with $x_1 > x_2 > ... > x_n$. Let p be a prime let $h = \gcd(\phi_p(a), \phi_p(b)) \in \mathbb{Z}_p[x_1, ..., x_n]$. If p does not divide $LC(a)$ then
>
> (i) $LM(h) \geq LM(g)$ and
>
> (ii) if $LM(h) = LM(g)$ then $\phi_p(g)|h$ and $h|\phi_p(g)$.

(c) (30 marks)

Implement the modular GCD algorithm of section 7.4 in Maple. Implement two subroutines, subroutine MGCD that computes the GCD modulo a sequence of primes (use 4 digit primes), and subroutine PGCD that computes the GCD at a sequence of evaluation points (use $0, 1, 2, ...$ for the evaluation points). Note, subroutine PGCD is recursive. Test your algorithm on the following example polynomials in $\mathbb{Z}[x, y, z]$. Use x as the main variable. First evaluate out z then y.

```
> c := x^3+y^3+z^3+1; d := x^3-y^3-z^3+1;
> g := x^4-123454321*y*z^2*x^2+1;
> MGCD(c,d,[x,y,z]);
> MGCD(expand(g*c),expand(g*d),[x,y,z]);
> MGCD(expand(g^2*c),expand(g^2*d),[x,y,z]);

> g := z*y*x^3+1; c := (z-1)*x+y+1; d := (z^2-1)*x+y+1;
> MGCD(expand(g*c),expand(g*d),[x,y,z]);
> g := x^4+z^2*y^2*x^2+1; c := x^4+z*y*x^2+1; d := x^4+1;
> MGCD(expand(g*c),expand(g*d),[x,y,z]);
> g := x^4+z^2*y^2*x^2+1; c := z*x^4+z*x^2+y; d := z*x^4+z^2*x^2+y;
> MGCD(expand(g*c),expand(g*d),[x,y,z]);
```

Please make your MGCD procedure print out the sequence of primes it uses using `printf(" p=%d\n",p);` .

Please make your PGCD procedure print out the sequence of evaluation points α that it uses for each variable u using `printf("    %a=%d\n",u,alpha);`

In PGCD you **MUST** compute mod p. You may use the `Content(...) mod p`, `Primpart(...) mod p`, `Interp(...) mod p` and `Divide(...) mod p` commands and `Gcd(...) mod p` for computing univariate gcds over $\mathbb{Z}_p$.

Note, procedures MGCD and PGCD on pages 307 and 309 in Chapter 7 of the Geddes text do not identify unlucky primes and unlucky evaluation points correctly.

Question 2: Sparse Interpolation Algorithms

(a) (5 marks) Apply Ben-Or/Tiwari sparse interpolation to interpolate

$$f(w,x,y,u) = 101w^5x^3y^2u + 103w^3xy^3u^2 + 107w^2x^5y^2 + 109x^2y^3u^5$$

over $\mathbb{Q}$ using Maple. You will need to compute the integer roots of the $\lambda(z)$ polynomial and solve a linear system over $\mathbb{Q}$.

Now it is very inefficient to run the algorithm over $\mathbb{Q}$. Repeat the method modulo a prime p, i.e., interpolate f modulo p. Assume you know that $\deg f < 16$. Pick p suitably large so that you can recover all monomials of total degree $d \leq 15$. See the `Roots(...) mod p` and `Linsolve(...) mod p` commands.

(b) (5 marks)

REFERENCE (a copy is available on the course web page):
Michael Ben-Or and Prasoon Tiwari. A deterministic algorithm for sparse multivariate polynomial interpolation. *Proc. STOC '88*, ACM press, 301-309, 1988.

The Ben-Or/Tiwari sparse interpolation algorithm interpolates a polynomial $f(x_1, x_2, \ldots, x_n)$ in two main steps. First it determines the monomials then it solves a linear system for the unknown coefficients of the polynomial. Let

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{t} a_i M_i$$

where a_i are the coefficients and M_i are the monomials. Let $a = [a_1, a_2, \ldots, a_t]$ be the vector of unknown coefficients. Let $v = [v_0, v_1, \ldots, v_{t-1}]$ be the vector of values where $v_j = f(2^j, 3^j, 5^j, \ldots, p_n^j)$. Let $m_i = M_i(2, 3, 5, \ldots, p_n)$ be the value of the monomial M_i. The linear system to be solved is $V^T a = v$ where

$$V^T = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ m_1 & m_2 & m_3 & \ldots & m_t \\ m_1^2 & m_2^2 & m_3^2 & \ldots & m_t^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m_1^{t-1} & m_1^{t-1} & m_1^{t-1} & \ldots & m_t^{t-1} \end{bmatrix}$$

is a transposed Vandermonde matrix. Solve this linear system for the problem in part (a) using the $O(t^2)$ method.

(c) (5 mark bonus) Suppose $f(w, x, y, z)$ is a polynomial in $\mathbb{Z}[w, x, y, z]$. Without first interpolating $f(w, x, y, z)$ is there a fast way to get a bound on the **total degree** of $f(w, x, y, z)$ using evaluation and univariate interpolation? State any assumptions you need.