Part d and e

October 16, 2023 11:49 AM

Part (d).

$$A_{ij} := \frac{A_{kk} \cdot A_{ij} - A_{ik} A_{kj}}{A_{k-1k-1}}$$

The polynomials in A have degree d.
At step $k=1$ we do two polynomial mults of degree d by d then divide them by 1 (polynomial of degree $0 = (k-1)d$) to get

$\deg(A_{ij}) = 2d$ for $2 \le i,j \le n \Rightarrow (n-1)^2$ times.

At step $k=2$ we do two poly mults of degree $2d \times 2d$ then divide by A_{11} of degree $d = (k-1)d$ to get

$\deg(A_{ij}) = 3d$ for $3 \le i,j \le n. \Rightarrow (n-2)^2$ times.

In the third elimination step $(k=3)$ we multiply two polynomials of degree $3d \times 3d$ then divide by A_{22} of degree $2d$ to get

$\deg(A_{ij}) = 4d$ for $4 \le i,j \le n$. so $(n-3)$ times

Thus at step k we do two polynomial mults of degree kd by kd then divide by them A_{kk} of degree $(k-1)d$ to get

$\deg(A_{ij}) = kd + kd - (k-1)d = (k+1)d$.

Multiplying two polynomials of degree kd by kd does $(kd+1)^2$ mults in F.
Dividing two polynomials of degree $n = 2kd$ by $m = (k-1)d$ does

$m(n-m+1)$ mults in F

$= (k-1)d\left[2kd - (k-1)d + 1\right]$

$= (k-1)d\,(kd+d+1)$.

The total # multiplications in F is

$$\sum_{k=1}^{n-1} (n-k)^2 \left[2(kd+1)^2 + (k-1)d(kd+d+1)\right]$$

$= \frac{1}{10} n^5 d^2 +$ terms of lower degree (by Maple)

$\in O(n^5 d^2)$.

Part (e).

If $\deg(A_{ij}) = d$ for $1 \le i,j \le n$ then $\deg(\det(A)) \le nd$.

In the worst case where $\deg(\det(A)) = nd$ we need $nd+1$ points to interpolate $\det(A)$. Newton interpolation does $\frac{3}{2}n^2 - \frac{3}{2}n$ mults in F to interpolate a polynomial of degree n (See notes). So to interpolate $\det(A)$ Newton does $\frac{3}{2}(nd+1)^2 - \frac{3}{2}(nd+1)$ mults in F.

We need to evaluate n^2 polynomials A_{ij} of degree d at $nd+1$ points. Horner does d mults for 1 evaluation so $(nd+1)n^2 \cdot d$ in total.

To compute $\det(A)$ for $A_{ij} \in F$ a field, Gaussian elimination on an $n \times n$ matrix A does

$$\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n \text{ mults in } F \text{ to triangularize } A.$$

To compute $\det(A)$ we need to multiply $A_{11} \cdot A_{22} \cdots A_{nn}$ so $n-1$ mults for a total of $\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{7}{6}n - 1$. We do this $nd+1$ times.

The total # mults in F is

$$\underbrace{(nd+1)n^2 d}_{\text{evaluation}} + \underbrace{(nd+1)\left(\tfrac{1}{3}n^3 - \tfrac{1}{2}n^2 + \tfrac{7}{6}n - 1\right)}_{\text{determinants in } F} + \underbrace{\tfrac{3}{2}(nd+1)^2 - \tfrac{3}{2}(nd+1)}_{\text{interpolation}}$$

$$= (n^3 d^2 + \cdots) + (\tfrac{1}{3}n^4 d + \cdots) + (\tfrac{3}{2}n^2 d^2 + \cdots)$$

$$= n^3 d^2 + \tfrac{1}{3}n^4 d + \cdots$$

$$\in O(n^3 d^2 + n^4 d) \text{ mults in } F.$$

This is two orders of magnitude faster than Bareiss/Edmonds which does $O(n^5 d^2)$ mults in F.

Notice that the interpolation cost is negligible for large n, m.