A1Q3 Solution, Copyright Michael Monagan

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(a) Using $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$ we have.

(i) $|cf(n)| \leq |c|\,|f(n)|$ for all $n \geq 1 \Rightarrow cf(n) \in O(f(n))$

(ii) $|f(n)| \leq \frac{1}{|c|} \cdot |cf(n)|$ for all $n \geq 1 \Rightarrow f(n) \in O(cf(n))$

Therefore $O(cf(n)) = O(f(n))$ for $c > 0$.

Note the result and proof hold for $c < 0$ but not $c = 0$.

(b). Show that $O(\log_a n) = O(\log_b n)$.

$\log_a n = \frac{1}{\ln a} \cdot \ln n \Rightarrow O(\log_a n) = O\left(\frac{\ln n}{\ln a}\right) = O(\ln n)$ by part (a).

$\log_b n = \frac{1}{\ln b} \cdot \ln n \Rightarrow O(\log_b n) = O\left(\frac{\ln n}{\ln b}\right) = O(\ln n)$ by part (a).

Therefore $O(\log_a n) = O(\log_b n)$.

(c) (i) $2n\,O(2n+1) = O(4n^2+2n) = O(4n^2) = O(n^2)$.

(ii) $O(2(n+1)^2 + 3n) = O(2n^2 + 4n + 2 + 3n) = O(n^2)$

(iii) $O(n^2) + n\,O(n^2/3) = O(n^2) + O(n^3/3) = O(n^2) + O(n^3)$
$= O(n^2 + n^3) = O(n^3)$.

(iv) $O(2^n + n^3) = O(2^n)$ since $2^n > 1 \cdot n^3 \;\forall\, n \geq 10$.