Interpolation in $F[y]$.

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Suppose $A, B \in F[x,y]$ where F is a field.

Suppose $deg(A,x) = deg(B,x) = dx$ and
$deg(A,y) = deg(B,y) = dy$.

How many multiplications in F does it take to compute
$C = A \cdot B$?

Let $A = \sum_{i=0}^{dx} \sum_{j=0}^{dy} \underbrace{a_{ij}}_{\in F} y^j x^i \quad B = \sum_{i=0}^{dx} \sum_{j=0}^{dy} b_{ij} y^j x^i$

$(3x^2y + 5x + 2y + 5)(2x^2 - 5y^2 + 3xy + 2x)$.

The classical algorithm multiplies $a_{ij} \cdot b_{k,l}$ so
$\leq [(dx+1)(dy+1)][(dx+1)(dy+1)] \in O(dx^2 dy^2)$ mults in F.

$$
\begin{array}{ccc}
\text{Consider } \overset{dy\ \ dy}{A,B} \in F[y][x] & \xrightarrow[O(dx^2dy^2)]{\text{classical}} & C = \overset{2dy}{A \cdot B} \in F[y][x] \\
\Big\downarrow y = \alpha_j,\ j = 0,1,\dots,2dy & & \Big\uparrow \text{Interpolate } y.\ (3) \\
(1)\ a_j = A(x,\alpha_j) \in F[x] & \xrightarrow[\leq (dx-1)(dx+1)\ (2)]{\text{class alg.}} & C_j = a_j(x) \cdot b_j(x). \\
b_j = B(x,\alpha_j) & & \text{deg} \leq dx.
\end{array}
$$

Cost ?? $A = \boxed{a_0(y)} \cdot x^0 + \boxed{a_1(y)} \cdot x^1 + \cdots + \boxed{a_{dx}(y)} \cdot x^{dx}$ (each of degree dy)

$B = \boxed{b_0(y)} x^0 + \boxed{b_1(y)} x^1 + \cdots + \boxed{b_{dx}(y)} \cdot x^{dx}$

(1) Evaluations: $\underset{\#\text{ polys in } y}{2(dx+1)} \cdot \underset{\text{Horner}}{dy} \cdot \underset{\#\alpha\text{'s}}{(2dy+1)} \in O(dx\,dy^2)$.

(2) Mults in $F[x]$: $\leq \underset{\text{each } x}{(dx+1)^2} \cdot \underset{\#\alpha\text{'s}}{(2dy+1)} \in O(dx^2 dy)$.

(3) Interpolation: $C = \boxed{c_0(y)} \cdot x^0 + \boxed{c_1(y)} x^1 + \cdots + \boxed{c_{2dx}(y)} x^{2dx}$

$\underset{\#\ c_i(y)\text{'s}}{(2dx+1)} \cdot \Big[O(\underset{\text{Lagrange or Newton}}{(2dy+1)^2}) = O(4dy^2 + 4dy + 1) = O(dy^2) \Big]$

$= O(dx\,dy^2)$.

Total # mults. in F is $O(dx\,dy^2) + O(dx^2 dy) + O(dx\,dy^2)$

$= O(dx\,dy^2 + dx^2 dy)$

Total # mults. in F is $O(dx\,dy^2) + O(dx^2 dy) + O(dx\,dy^2)$
$= O(2dx\,dy^2 + dx^2 dy)$
$= O(\cancel{2}dx\,dy^2) + O(dx^2 dy)$
$= O(dx\,dy^2 + dx^2 dy)$.

This is a "cubic" algorithm.

$\mathbb{Z}_p[y,x]$. $\underline{A \cdot B} = C$.

$dx = 1000, 2000, 4000, 8000$
$dy = 1000, 2000, 4000, 8000$

$(8000)^4 = 2^{12} \cdot 10^{12} = 4 \cdot 10^{15}$
$(8000)^3 = 2^9 \cdot 10^9 < 10^{12}$.

Newton's Method

$$f(x) = v_1 + v_2(x-\alpha_1) + \cdots + v_n(x-\alpha_1)\cdots(x-\alpha_{n-1}).$$

$$y_k = f(\alpha_k) = \overbrace{v_1 + v_2(\alpha_k-\alpha_1) + v_3(\alpha_k-\alpha_1)(\alpha_k-\alpha_2) + \cdots}^{S} + v_k(\alpha_k-\alpha_1)\cdots(\alpha_k-\alpha_{k-1}).$$

$p=1$ $\quad p = p\cdot(\alpha_k-\alpha_1)$ $\quad p = p\cdot(\alpha_k-\alpha_2)$ $\quad \cdots\cdots$ $\quad p = p\cdot(\alpha_k-\alpha_{k-1})$.

for $k = 1, 2, \ldots, n$ // compute v_k.
  $p = 1$. $S = 0$.
  for $i = 1, 2, \ldots, k-1$ do $S = S + v_i \cdot p$. $p = p\cdot(\alpha_k - \alpha_i)$.
  $v_k = (y_k - S)/p$.

mults $\displaystyle\sum_{k=1}^{n} 2(\underline{k-1}) = 2\boxed{\sum_{k=0}^{n-1} k} = 2\,\frac{(n-1)n}{2} = n^2 - n$.

$$f = v_1 + v_2(x-\alpha_1) + \cdots + v_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1})$$
$$= v_1 + (x-\alpha_1)[v_2 + (x-\alpha_2)[v_3 + \cdots + (\underbrace{v_{n-1} + (x-\alpha_{n-1})v_n}_{f})\cdots]]$$

$f = v_n$ $\quad d = 0$ $\quad i = 1$.
for k from $n-1$ by -1 to 1 do
  $f = v_k + (x-\alpha_k)\cdot f = v_k + \underbrace{x\cdot f}_{0 \text{ mults}} - \underbrace{\alpha_k \cdot f}_{1 \text{ mult.}}$ $\quad d = d+1$ $\quad i = i+1$.
  $\deg f = d$
output f.

mults $\displaystyle\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$.

Total cost of Newton is $n^2 - n + \frac{1}{2}n^2 - \frac{1}{2}n$
$= \frac{3}{2}n^2 - \frac{3}{2}n \in O(n^2)$ mults in F.

Lagrange: $= \boxed{?}\cdot n^2 + \boxed{}\cdot n + \boxed{}$.

Lagranges Method.

① Compute $L(x) = \prod_{i=1}^{n} (x - \alpha_i)$ Master Polynomial.

② Compute $L_i(x) = \dfrac{L(x)}{x - \alpha_i}$ for $1 \le i \le n$.

$$
\begin{array}{r l l}
 & \bullet x^{n-1} & \\
1 \cdot x - \alpha_i \; \big) & \bullet x^{n} + \bullet x^{n-1} + \cdots + \bullet & \\
 & -(\bullet x^{n} + \bullet x^{n-1}) & 1 \text{ mult.} \\
\hline
 & 0 x^{n} + \bullet \cdot x^{n-1} + \cdots + \bullet & \\
 & \vdots & \\
 & \bullet x + \bullet & \leftarrow \deg 1. \\
\hline
 & \text{remainder } \bullet & \leftarrow \deg = 0.
\end{array}
$$

division steps $\le n$. Each does 1 mult. in F.
So $\le n$ mults in F.

③ Compute $a_i = y_i / L_i(\alpha_i)$

④ $f(x) = \sum_{i=1}^{n} \overset{F}{a_i} \cdot L_i(x)$.

scalar x.