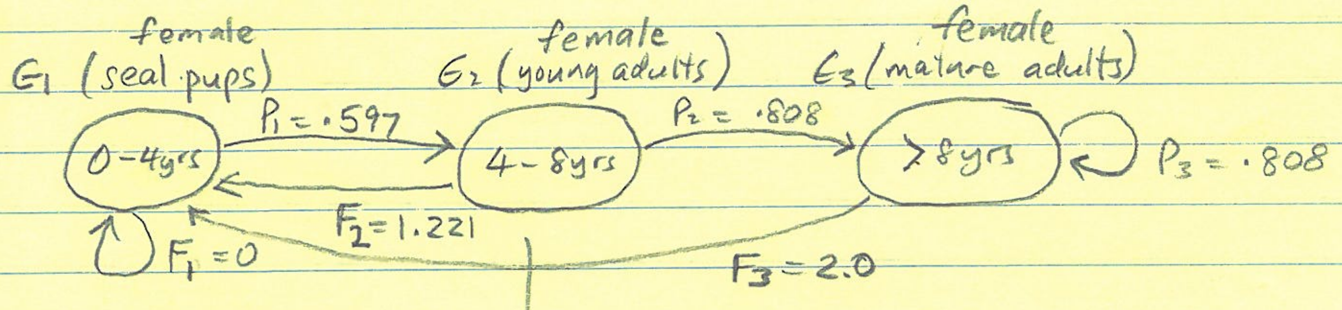
The Leslie Age Distribution Model

Suppose we divide the females of a population into n age groups, $G_1, G_2, \ldots, G_n$. Let F_i be the fertility rate of group G_i.

F_i = the # of <u>females</u> born in X years to an individual.

Let P_i be the probability an individual in group G_i survives X years (survival rates).



Let N_{it} be the number of females in G_i at time t.
The vector $N_t = [N_{1t}\ N_{2t}\ \cdots\ N_{n,t}]$ is called the pop. vector at time t. The model says,

after X years

$$\underbrace{\begin{bmatrix} N_{1,t+1} \\ N_{2,t+1} \\ N_{3,t+1} \end{bmatrix}}_{N_{t+1}} = \begin{bmatrix} F_1 N_{1t} + F_2 N_{2t} + F_3 N_{3t} \\ P_1 N_{1t} \\ P_2 N_{2t} + P_3 N_{3t} \end{bmatrix} = \underbrace{\begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & P_3 \end{bmatrix}}_{\text{Leslie matrix } L} \underbrace{\begin{bmatrix} N_{1t} \\ N_{2t} \\ N_{3t} \end{bmatrix}}_{N_t}$$

I.e. $N_{t+1} = L N_t$. Let N_0 be the initial pop at time $t=0$.

$N_1 = L N_0$ = pop after 4 years

$N_2 = L N_1 = L^2 N_0$ = pop after 8 years

$N_{20} = L N_{19}$ = the pop after 80 years.

For $N_0 = [1,0,0]$. Using Maple I get

$N_{19} = [496, 200, 241]$ and $N_{20} = [734, 296, 356]$

Let $D_t = N_t/(N_{1t} + \cdots + N_{nt})$ be the pop. distribution vector at time t.

$D_{19} = [.529431, .213607, .256960]$ $\Rightarrow$ The pop. distribution

$D_{20} = [.529431, .213607, .256960]$ has stabilized!

Notice $\frac{734}{496} = 1.475$ $\frac{296}{200} = 1.475$ and $\frac{356}{241} = 1.475$

The pop. of all age groups is now increasing by 47.5% / 4 years. After a short time ($20 \times 4 = 80$ years) we have

$$\underbrace{L \begin{bmatrix} .529 \\ .213 \\ .257 \end{bmatrix}}_{V} = \underbrace{1.475}_{\lambda} \underbrace{\begin{bmatrix} .529 \\ .213 \\ .257 \end{bmatrix}}_{V}$$

pop distribution

i.e. V an eigenvector of L with eigen value $\lambda = 1.47968$.

= how fast pop increases

Theorem. If L is a Leslie matrix and $0 < P_i \leq 1$ and at least one $F_i > 0$ then L has one positive eigenvalue λ^+ called the dominant eigenvalue of L. (see section 5.8) and if N_0 is a non-zero pop. vector then

$\lim_{t \to \infty} D_t = V$ where $LV = \lambda^+ V$ and V is a prob. vector.

eigenvector of L

2 complex eigenvalues

$L = \begin{bmatrix} 0 & 1.221 & 2.0 \\ .597 & 0 & 0 \\ 0 & .808 & .808 \end{bmatrix}$ $\det(L - \lambda I) = (\lambda - 1.475)(\lambda^2 + .67\lambda + .255)$

$\lambda^+ = 1.475$

Solve $(L - \lambda^+ I)V = 0 \Rightarrow V = \begin{bmatrix} .529 \\ .214 \\ .257 \end{bmatrix}$

What would happen if the survival rates P_1, P_2, P_3 decreased by 50%? [Perhaps an increase in sharks]

$$L \to \begin{bmatrix} 0 & 1.221 & 2.0 \\ .298 & 0 & 0 \\ 0 & .404 & .404 \end{bmatrix}$$

I get $\lambda^+ = .913$.
$\lambda^+ = .913$ means the pop. is decreasing.
by $1 - .913 = 8.7\% / 4$ years.

If $\lambda^+ > 1$ the pop. increases exponentially.
If $\lambda^+ < 1$ the pop. declines exponentially.
If $\lambda^+ = 0$ the pop. is "stable".

How healthy is the seal population? λ^+ is one measure.
What proportion P_1 of the seal pups need to survive for the pop. to survive i.e. $\lambda^+ = 1$.

fix $\lambda^+ = 1$

$$\text{Let } L = \begin{bmatrix} 0 & 1.221 & 2.0 \\ P_1 & 0 & 0 \\ 0 & .808 & .808 \end{bmatrix}$$

Solve $\boxed{Lv = 1 \cdot v}$ for P_1

$\Rightarrow Lv - Iv = 0$

$\Rightarrow (L - I)v = 0$

$\underset{v \neq 0}{\Rightarrow} \boxed{\det(L - I) = 0}$

solve for P_1

$$L - I = \begin{bmatrix} -1 & 1.221 & 2.0 \\ P_1 & -1 & 0 \\ 0 & .808 & -.192 \end{bmatrix}$$

$\det(L - I) = 1.855 P_1 - .192 = 0$

$\Rightarrow P_1 = .104$

[The pop. is very healthy]

[How well does the model approximate the population?]