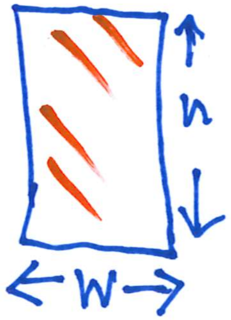
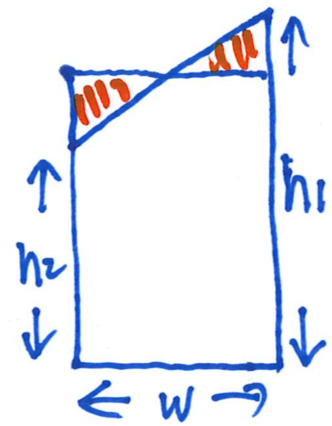
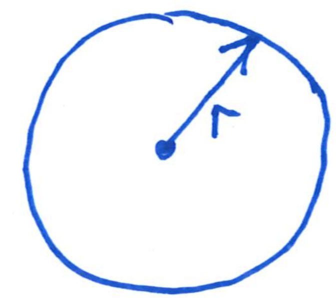
5.1 Areas and Distances



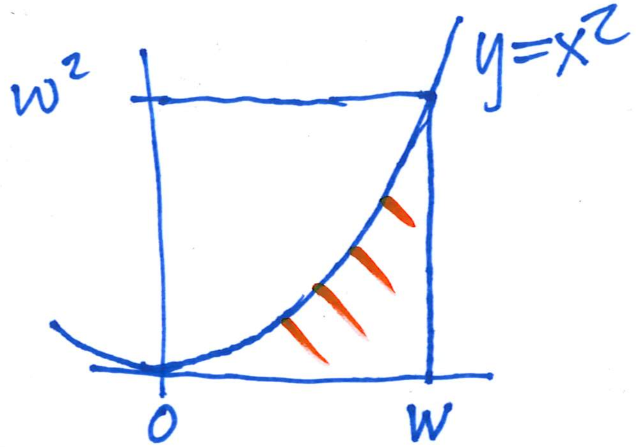
$\text{Area} = h \cdot w$

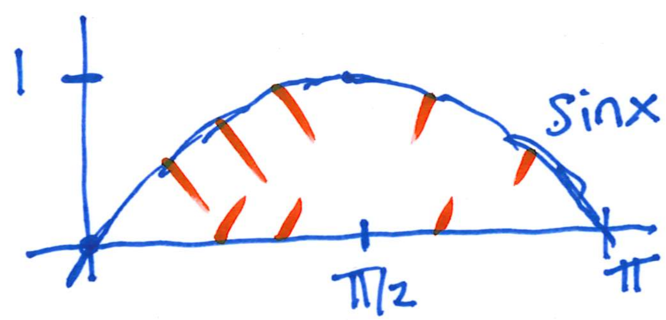


$\text{Area} = \frac{h_1 + h_2}{2} \cdot w$



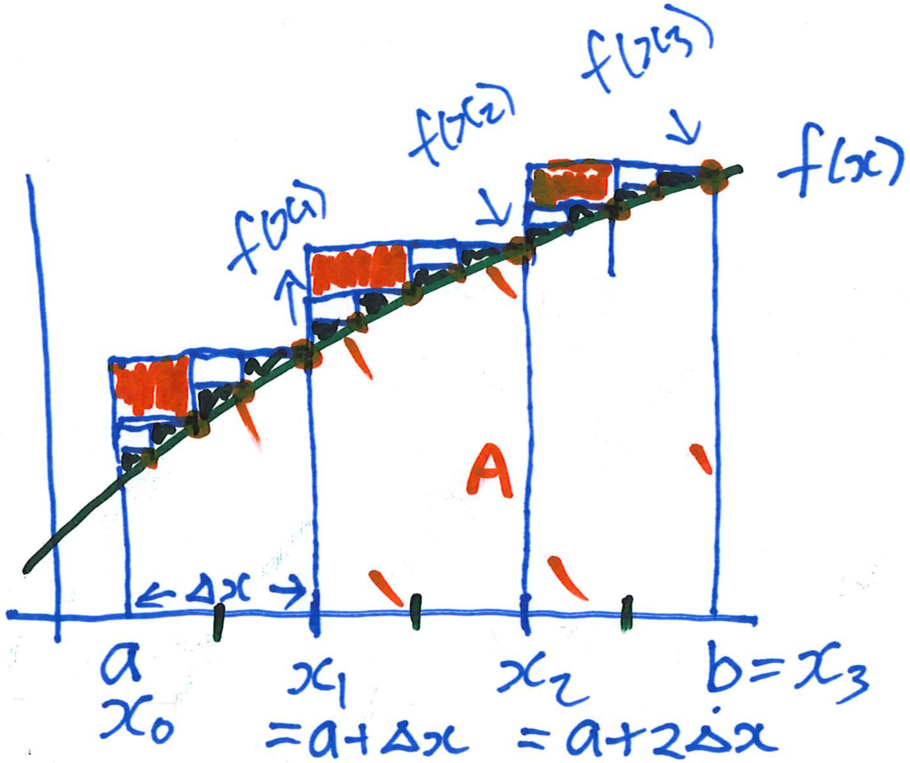
$\text{Area} = \pi r^2$





Area = ?

Let A be te area under $f(x)$ between $x=a$ and $x=b$.



Divide $[a,b]$ into n subintervals $[x_0,x_1]$, $[x_1,x_2]$, ..., $[x_{n-1},x_n]$ of equal width $\Delta x=(b-a)/n$ so $x_i=a+i\Delta x$

Approximate A by n rectangles

$$R_n=\Delta x\cdot f(x_1)+\Delta x f(x_2)+\cdots+\Delta x f(x_n)$$
$$=\Delta x\left[f(x_1)+f(x_2)+\cdots+f(x_n)\right]$$

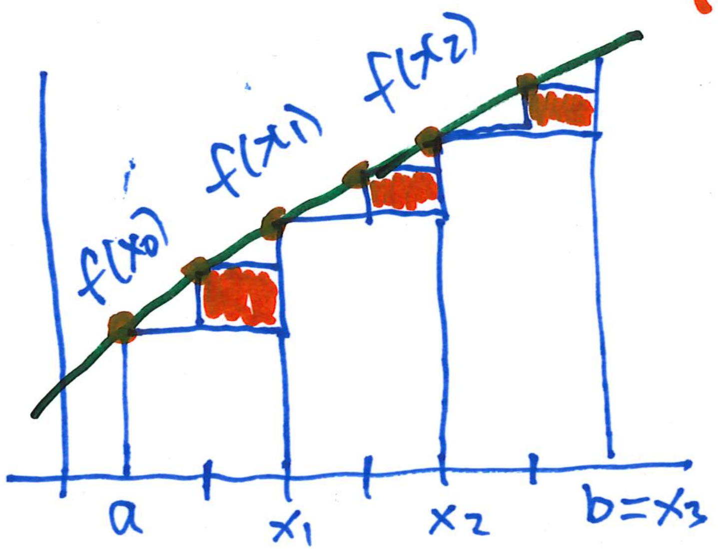
$n=3$

$n=6$

$n=12$

$$\boxed{A=\lim_{n\to\infty} R_n} \qquad \boxed{A=\lim_{n\to\infty} L_n}$$

left rect. rule.



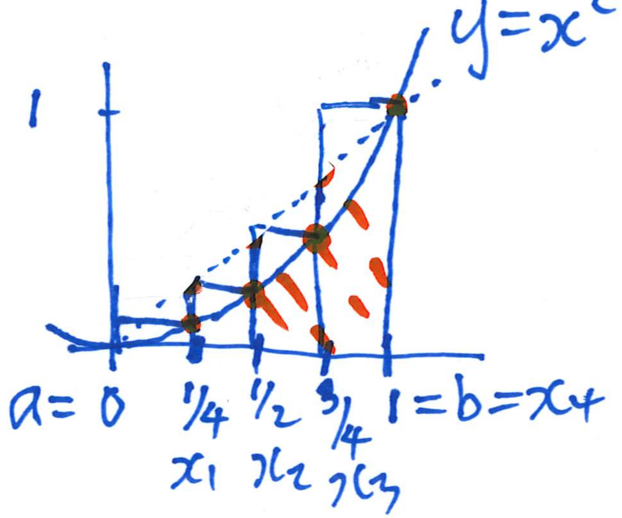
$n=3$

$$L_n=\Delta x f(x_0)+\Delta x f(x_1)+\cdots+\Delta x f(x_{n-1})$$
$$R_n=\Delta x f(x_1)+\Delta x f(x_2)+\cdots+f(x_n)\Delta x$$

right rectangle rule

$L_n<A<R_n$ because $f(x)$ is increasing.

Example



$n=4$

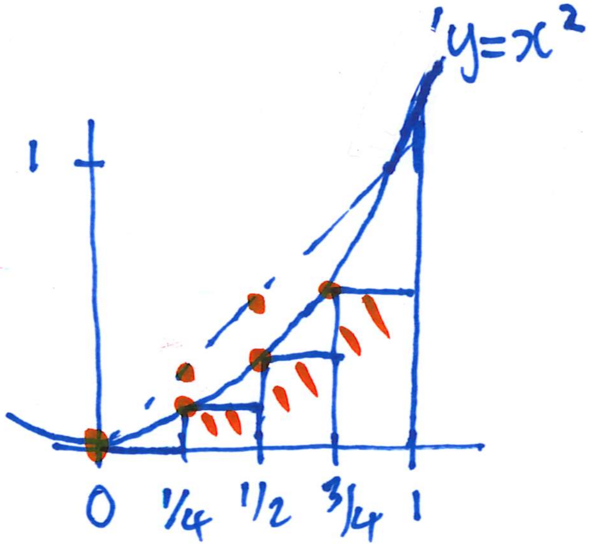
$\Delta x = \frac{1}{4}$

$$R_4 = \frac{1}{4}\left(f(\tfrac{1}{4}) + f(\tfrac{1}{2}) + f(\tfrac{3}{4}) + f(1)\right)$$

$$= \frac{1}{4}\left((\tfrac{1}{4})^2 + (\tfrac{1}{2})^2 + (\tfrac{3}{4})^2 + 1^2 \right)$$

$$= \frac{1}{4}\left(\frac{1 + 4 + 9 + 16 = 30}{16} \right) = \frac{30}{64} = .46875$$

$$R_{1000} = .33383$$



$$L_4 = \frac{1}{4}\left(f(0) + f(\tfrac{1}{4}) + f(\tfrac{1}{2}) + f(\tfrac{3}{4})\right)$$

$$= \frac{1}{4}\left(0 + (\tfrac{1}{4}) + (\tfrac{1}{2})^2 + (\tfrac{3}{4})^2\right)$$

$$= \frac{1}{4}\left(\frac{0 + 1 + 4 + 9 = 14}{16} \right) = \frac{14}{64} = .21875$$

$$L_{1000} = .33283$$

$$.33283 = L_{1000} < A < R_{1000} = .33383$$

Recall $1^2+2^2+3^2+\cdots+n^2 = \sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1) = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$

$f(x)=x^2 \quad a=0,\ b=1 \quad \Delta x = (b-a)/n = \frac{1}{n} \quad x_i = a+i\Delta x = \frac{i}{n}.$

$$R_n = \frac{1}{n}\left(f\left(\tfrac{1}{n}\right)+f\left(\tfrac{2}{n}\right)+f\left(\tfrac{3}{n}\right)+\cdots+f\left(\tfrac{n}{n}\right)\right)$$
$$= \frac{1}{n}\left(\left(\tfrac{1}{n}\right)^2+\left(\tfrac{2}{n}\right)^2+\cdots+\left(\tfrac{n}{n}\right)^2\right)$$
$$= \frac{1}{n^3}\left(\underline{1^2+2^2+3^2+\cdots+n^2}\right) = \frac{1}{n^3}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right) = \frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}$$

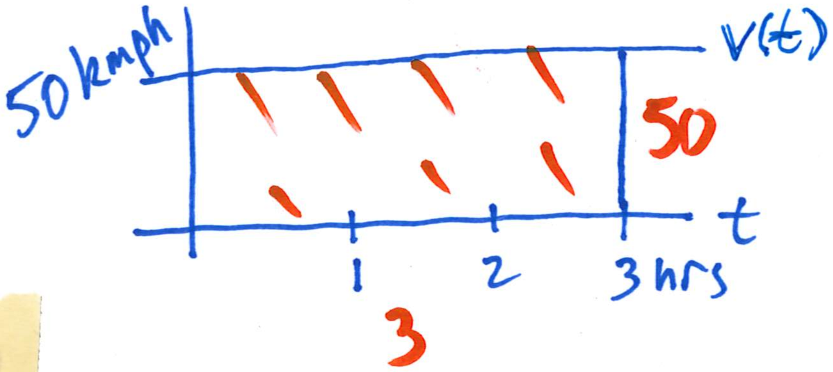
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\frac{1}{3}+\underbrace{\frac{1}{2n}}_{\to 0}+\underbrace{\frac{1}{6n^2}}_{\to 0}\right) = \frac{1}{3} = .3333\cdots$$

$$L_n = \frac{1}{n}\left(f\left(\tfrac{0}{n}\right)+f\left(\tfrac{1}{n}\right)+\cdots+f\left(\tfrac{n-1}{n}\right)\right) = \frac{1}{n}\left(0^2+\left(\tfrac{1}{n}\right)^2+\left(\tfrac{2}{n}\right)^2+\cdots+\left(\tfrac{n-1}{n}\right)^2\right)$$
$$= \frac{1}{n^3}\left(0+\underbrace{1^2+2^2+\cdots+(n-1)^2+n^2}_{=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}}-n^2\right)$$
$$= \frac{1}{n^3}\left(\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}\right)$$
$$= \frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}$$

$$\lim_{n\to\infty} L_n = \frac{1}{3}. \quad ☺$$

The Distance Problem

Let $v(t)$ be the velocity of my car.

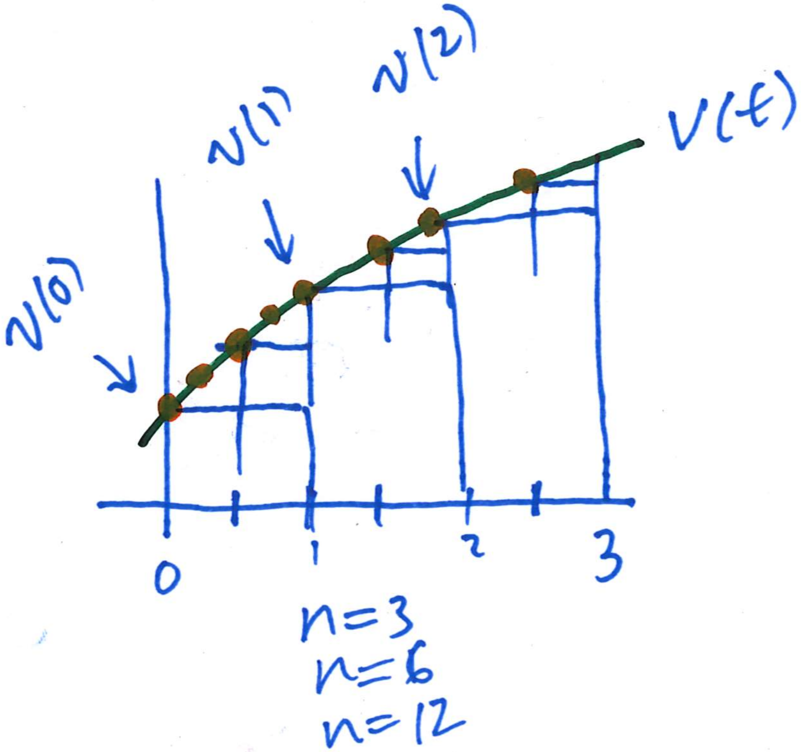


How far do I go? 150 km.

$150 =$ the area under $v(t)$!

Let D be the distance travelled on $[a,b]$. Then

$$D = \text{Average Velocity} \times (b-a).$$



$$L_3 = 1\cdot v(0) + 1\cdot v(1) + 1\cdot v(2) . < D$$

approx the distance travelled on the subintervals

$$\lim_{n\to\infty} L_n = D \; ?$$

$$\lim_{n\to\infty} L_n = \text{Area under } v(t) \text{ on } [a,b].$$

$$\Rightarrow \boxed{D = \text{Area under } v(t).}$$