Lec24B Logarithmic Case: Logarithmic Part Michael Monagan

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Theorem 12.7 (Trager-Rothstein – Logarithmic case)

$\int \frac{C(\theta)}{D(\theta)} dx$ $\quad C, D \in F[\theta]$, $\theta = \log u$, $u \in F$, $\theta \notin F$, $\theta' \neq 0$
$0 \le \deg_\theta C < \deg_\theta D$, $\mathrm{lc}_\theta D = 1$,
$\gcd(C, D) = 1$, $\gcd(D, D') = 1$ in $F[\theta]$.

Let $R(z) = \mathrm{res}_\theta(C - z D', D) \in F[z]$
$\qquad\qquad\qquad\qquad \underset{F[\theta]}{\cap}$

(i) $\int \frac{C}{D} dx$ is elementary iff all roots of $R(z)$ are constants

(ii) If $\int \frac{C}{D} dx$ is elementary then $\int \frac{C}{D} dx = \sum c_i \log v_i$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underset{\mathbb{C}}{\cap}$

where c_i are the distinct roots of $R(z)$
and $\underline{v_i} = \text{monic } \underline{\gcd(C - c_i D', D)} \in F[\theta]$

Example 1. $\int \frac{1}{\log x} dx = \int \frac{dx}{\theta}$ $\quad F(\theta) = \mathbb{C}(x)(\log x)$
$C = 1 \quad D = \theta \quad D' = \frac{1}{x}$

$R(z) = \mathrm{res}_\theta(1 - z\frac{1}{x}, \theta) = 1 - z\frac{1}{x} \in \mathbb{C}(x)[z]$.
$F[z][\theta]$ $\qquad$ constant in θ

$R(z) = 0 = 1 - z/x \Rightarrow z = x \Rightarrow c_1 = x \notin \mathbb{C}$
This means $\int \frac{1}{\log x} dx$ is not elementary.

Example 2. $\int \frac{1}{x \log x} dx = \int \frac{1/x}{1 \cdot \theta} dx$ $\quad C = 1/x$, $D = \log x$, $D' = \frac{1}{x}$

$F(\theta) = \mathbb{C}(x)(\log x)$.

$R(z) = \mathrm{res}_\theta(\underline{\tfrac{1}{x} - z\tfrac{1}{x}}, \theta) = \left(\frac{1}{x} - z\cdot\frac{1}{x}\right)^1$

$z = 1$ is a root of $R(z)$ so $c_1 = 1$. and $\int$ is elementary.

$v_1 = \gcd(\frac{1}{x} - 1\cdot\frac{1}{x}, \theta) = \gcd(0, \theta) = \cancel{1\cdot}\,\theta$

Therefore $\int \frac{1}{x \log x} dx = \underbrace{1 \cdot \log \theta}_{c_1 \cdot \log v_1} = \log \log x$.

Therefore $\int \frac{1}{x\log x}\,dx = 1\cdot\log\theta = \log\log x.$
$\qquad c_1 \cdot \log v_1$

Example 3. $\int \frac{(4-x)\log x - 3x}{x\log^2 x - x^2\log x}\,dx = \int \frac{\frac{4-x}{x}\theta - 3}{\theta^2 - x\theta}\,dx$

$F(\theta) = \mathbb{C}(x)(\log x)$

$C = \frac{4-x}{x} - 3$

$D = \theta^2 - x\theta$

$D' = 2\theta\cdot\frac{1}{x} - \theta - 1.$

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> alias(D=DD):
> f := ((4-x)*ln(x)-3*x)/(x*ln(x)^2-x^2*ln(x)): Int(f,x);
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$$\int \frac{(4-x)\ln(x) - 3x}{x\ln(x)^2 - x^2\ln(x)}\,dx$$

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> C := (4/x-1)*ln(x)-3; D := ln(x)^2-x*ln(x);
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$$C := \left(\frac{4}{x} - 1\right)\ln(x) - 3$$

$$D := \ln(x)^2 - x\ln(x)$$

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> Int( subs(ln(x)=theta,C/D), x ), theta=ln(x);
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$$\int \frac{\left(\frac{4}{x} - 1\right)\theta - 3}{\theta^2 - x\theta}\,dx,\ \theta = \ln(x)$$

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> TR := subs(ln(x)=theta, 'resultant'(C - z*diff(D,x),D,theta));
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$$TR := resultant\left(\left(\frac{4}{x} - 1\right)\theta - 3 - z\left(\frac{2\theta}{x} - \theta - 1\right), \theta^2 - x\theta, \theta\right)$$

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> factor(TR);
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$$(z-3)\,(z-1)\,(x-1)$$

$\gcd(A,B)$ $\mathbb{Z}[x,y,z,\cdots]$

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> c[1] := 3:
  v[1] := subs(ln(x)=theta,'gcdex'(C-c[1]*diff(D,x),D,theta));
  'c[1]' = c[1], 'v[1]' = v[1];
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$$v_1 := gcdex\left(\left(\frac{4}{x} - 1\right)\theta - \frac{6\theta}{x} + 3\theta, \theta^2 - x\theta, \theta\right)$$

$$c_1 = 3,\ v_1 = \theta$$

gcdex monic gcd in $F[\theta]$.
$\qquad\qquad\qquad\qquad \mathbb{Q}(x)$.

```
> c[2] := 1:
  v[2] := subs(ln(x)=theta,'gcdex'(C-c[2]*diff(D,x),D,theta));
  'c[2]' = c[2], 'v[2]' = v[2];
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$$v_2 := gcdex\left(\left(\frac{4}{x} - 1\right)\theta - 2 - \frac{2\theta}{x} + \theta, \theta^2 - x\theta, \theta\right)$$

$$c_2 = 1,\ v_2 = \theta - x$$

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> Int(C/D,x) = c[1]*log(v[1])+c[2]*log(v[2]), theta=ln(x);
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$$\int \frac{\left(\frac{4}{x} - 1\right)\ln(x) - 3}{\ln(x)^2 - x\ln(x)}\,dx = 3\ln(\theta) + \ln(\theta - x),\ \theta = \ln(x)$$

$$\int \frac{\frac{4-x}{x}\theta-3}{\theta^2-x\theta} = \int \frac{\frac{3}{x}}{\theta} + \frac{\frac{1-x}{x}}{\theta-x} = \underset{\underset{\mathbb{Q}}{\Psi}}{c_1}\cdot\log\theta + \underset{\underset{\mathbb{Q}.}{\Psi}}{c_2}\log(\theta-x) + v_0(\theta) + \underset{\text{C.T.}}{\Sigma L}.$$

(*)

$F(\theta) = \mathbb{Q}(x)(\log x)$

PF
$$\frac{\theta\frac{4-x}{x}-3}{\theta^2-x\theta} = \frac{A}{\theta^1} + \frac{B}{\theta^1-x} \qquad A, B \in \mathbb{Q}(x).$$

$$\theta\left(\frac{4-x}{x}\right)-3 = A(\theta-x) + \underline{B\theta}$$

$$\big|\theta=0 \qquad -3 = A(-x)+0 \Rightarrow A = \frac{3}{x}$$

$$\big|\theta=x \qquad 1-x = A\cdot 0 + Bx \Rightarrow B = (1-x)/x$$

$$\Rightarrow \frac{\frac{3}{x}}{\theta} + \frac{\frac{1}{x}-1}{\theta-x} = \underset{\underset{3}{\shortparallel}}{c_1}\cdot\frac{1}{\theta}\cdot\frac{1}{x} + \underset{\underset{1}{\shortparallel}}{c_2}\frac{1}{\theta-x}\left(\frac{1}{x}-1\right) + \underline{v_0'} + \underline{\Sigma L'}$$

$$\int f = 3\cdot\log\theta + 1\cdot\log(\theta-x) = 3\log\log x + \log(\log x - x).$$