Lec23B Liouville's Principle Michael Monagan

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Theorem 12.4 (Liouville's Principle)

Let F be a differential field with an algebraically closed constant field K e.g. $K=\mathbb{C}$. Let $f \in F$. Suppose.
$\int f(x)\,dx \in G$ where G is an elementary extension of F
i.e. $G = F(\theta_1, \ldots, \theta_n)$ where θ_i is exp, log, algebraic over $F(\theta_1, \ldots, \theta_{i-1})$. Then $\exists\, v_0, v_1, \ldots, v_m \in F$ and constants $c_1, c_2, \ldots, c_m \in K$ such that

$$\int f\,dx = v_0 + c_1 \log v_1 + \cdots + c_m \log v_m$$

i.e. if $\theta_i \notin F$ then θ_i is a logarithm.

E.g. $$\int x e^x + \frac{2}{1+x}\,dx = \underbrace{(x-1)e^x}_{v_0 \in F} + \underbrace{2}_{c_1 \in \mathbb{C}} \log \underbrace{(1+x)}_{v_1 \in F}$$

$F = \mathbb{C}(x)(e^x)$ $\qquad$ $G = F(\theta_1 = \log(1+x))$

E.g. Suppose $\int f\,dx = x e^{x^2}$ and $f \in F$.
L.P. $\Rightarrow e^{x^2} \in F$ and e^{x^2} appears in f.

Differentiating both sides
$$f = [x e^{x^2}]' = 1 \cdot e^{x^2} + 2x^2 e^{x^2} = (1+2x^2)e^{x^2}$$
Theorem 12.3 says $\deg(f', e^{x^2}) = \deg(f', e^{x^2})$

Observation: The theorem is true for $F = \mathbb{C}(x)$.

From Ch 11, $f \in \mathbb{C}(x)$,

$$\underset{\substack{\cap \\ \mathbb{C}(x)}}{\int f} = \underset{\substack{\cap \\ \mathbb{C}[x]}}{P} + \underset{\substack{\cap \\ \mathbb{C}(x)}}{\frac{A}{B}} + \int \frac{C}{D} = \underbrace{\frac{PB+A}{B}}_{= V_0 \in \mathbb{C}(x)} + \sum_i \underset{\substack{= \\ C_i}}{\overset{\substack{\mathbb{C} \\ \cup}}{\alpha_i}} \log \overset{\substack{\mathbb{C}[x] \subset \mathbb{C}(x). \\ \cup}}{v_i}$$

Proof By induction on n, # of new extensions needed.

The modern proof just uses partial fractions.

Special case of $n=1$ logarithmic extension θ.

Suppose $\int f(x)dx \in F(\theta)$ where $\theta = \log u$, $u \in F$, $u' \neq 0$, $\theta \notin F$, $f \in F$. $\Rightarrow \int f(x)dx = \frac{a(\theta)}{b(\theta)}$ where $a, b \in F[\theta]$, $\gcd(a,b) = 1$, $lc_\theta\, b(\theta) = 1$.

To Prove L.T. says $\frac{a(\theta)}{b(\theta)} = V_0 + c_1 \cdot \theta$ where $V_0 \in F$, $c_1 \in K$.

We will prove $\deg_\theta(b(\theta)) = 0 \Rightarrow \frac{a}{b} \in F[\theta]$.

TAC suppose $\deg_\theta b > 0$.

Let $b = 1 \cdot \prod_{i=1}^{\ell} b_i^{r_i}$ be the monic irreducible factorization of b over F,

i.e. $b_i \in F[\theta]$, $lc(b_i) = 1$, $\deg_\theta b_i > 0$, b_i is irreducible over F.

Let $\int f(x)dx = \frac{a(\theta)}{b(\theta)} = a_0(\theta) + \sum_{i=1}^{\ell} \sum_{j=1}^{r_i} \frac{a_{ij}(\theta)}{b_i(\theta)^j}$ be the PFD of $\frac{a}{b}$.

Satisfying $a_0, a_{ij} \in F[\theta]$, $\deg_\theta a_{ij} < \deg_\theta b_i$, $\gcd(a_{ij}, b_i) = 1$.

Differentiating.

$$\underset{\substack{\cap \\ F}}{f(x)} = a_0' + \sum_{i=1}^{\ell} \sum_{j=1}^{r_i} \frac{a_{ij}'}{b_i^{\,j}} - j \frac{b_i'}{b_i^{\,j+1}} \cdot a_{ij} \qquad (*)$$

$$f(x) = \ldots \quad \sum_{i=1} \sum_{j=1} \ldots \frac{\ldots}{b_i^{j}} \ldots \frac{\ldots}{b_i^{j+1}}$$
$$\underset{F}{\uparrow}$$

where $a_0', a_{ij}', b_i' \in F[\theta]$ by Th 12.2. Since $f \in F$ not $F(\theta)$ all functions of $\theta = \log u$ on the right must cancel.

Consider $j \cdot a_{ij} \cdot b_i' / b_i^{j+1}$ where $b_i \in F[\theta]$ is irreducible. Since $lc_\theta(b_i) = 1$ then by Th 12.2 then $\deg_\theta b_i' = \deg_\theta(b_i) - 1$.
Since b_i is irreducible then $\gcd(b_i'(\theta), \underset{\uparrow \text{irreducible.}}{b_i(\theta)}) = 1$ in $F[\theta]$

Hence $\gcd(j \cdot a_{ij} b_i', b_i) = 1$.

Consider the term in (*) when $j = r_i$. It is

$$\frac{a_{ir_i}'}{b_i^{r_i}} - \frac{r_i a_{ir_i}}{b_i^{r_i+1}} \cdot b_i'$$

$$26 \neq \frac{7}{3^{3}} + \frac{2}{3^2} + \frac{1}{5^{2}} - \frac{2}{5}$$

$$\times 3^2 5^2 \quad 5^2 3^2 \cdot 26 = \frac{7 \cdot 5^2}{3} + 2 \cdot 5^2 + 3^2 \cdot 1 - 3^2 \cdot 5$$

There is exactly one term in (*) with denominator $b_i^{r_i+1}$, hence it cannot cancel out. Hence $r_i = 0$ for all i.
Therefore $\deg_\theta b_i = 0$.

Hence $\int f(x)\,dx = a(\theta) \in F[\theta]$.

$\overset{\prime}{\Rightarrow} \quad \underset{F}{\overset{}{f(x)}} = a'(\theta) \in F[\theta]$.

From Th 12.2 $a' \in F \Rightarrow a = c \cdot \theta + d$. for some $c \in K$ and $d \in F$.

Hence $\int f(x)\,dx = c\theta + d$ as required by L.T.