Lec23A The Risch Structure Theorem 12.2 Michael Monagan

April 6, 2021 11:49 AM

Consider $\int (\underbrace{xe^{x/2}+e^x}_{f})\,dx$

We first choose a differential field F_n with $f \in F_n$.
We will choose $F_1 = \mathbb{Q}(x)(\theta_1 = e^{x/2})$ so $f = x\theta_1 + \theta_1^2$
which is a polynomial in θ_1.

Let K be a field of constants e.g. $K = \mathbb{Q}, \mathbb{C}$.
Let $F_n = K(x)(\theta_1, \theta_2, \ldots, \theta_n)$ where $\theta_i' \neq 0$ and θ_i is
(i) exponential over $F_{i-1} \Rightarrow \theta_i = e^{w_i}$, $w_i \in F_{i-1}$ OR
(ii) logarithmic over $F_{i-1} \Rightarrow \theta_i = \log u_i$, $u_i \in F_{i-1}$ OR
(iii) algebraic over $F_{i-1} \Rightarrow \exists p \in F_{i-1}[z]$ s.t. $p(\theta_i) = 0$.
[F_n is an elementary extension of $K(x)$].

Let $f \in F_n$. To compute $\int f(x)dx$ we want a representation F_n s.t.
(i) $\theta_i \notin F_{i-1} = K(x)(\theta_1, \ldots, \theta_{i-1})$ [for correctness]
(ii) θ_i is not algebraic over F_{i-1} [for efficiency]

Example $f = xe^{x/2} + e^x = x\theta_1 + \theta_2$

$F_2 = \mathbb{Q}(x)(\theta_1 = e^{x/2}, \theta_2 = e^x)$ $\theta_2 = \theta_1^2 \in F_1$. ×
$F_1 = \mathbb{Q}(x)(\theta_1 = e^{x/2})$.

$\deg(\theta_2' - \theta_1^2 + \theta_1, \theta_2) = 1$?
$= 0 + \theta_1$

$F_2 = \mathbb{Q}(x)(\theta_1 = e^x, \theta_2 = e^{x/2})$ $\theta_2 = \sqrt{\theta_1} \notin F_1$.
algebraic ext. $p(z) = z^2 - \theta_1$

algebraic ext.

$p(z) = z^2 - \theta_1$

$p(\theta_2) = \theta_2^2 - \theta_1 = 0.$

$$F_1 = \mathbb{Q}(x)(\theta_1 = e^{x/2})$$

$$f = xe^{x/2} + e^x = x \cdot \theta_1 + \theta_1^2.$$

Theorem 12.1 (the Risch Structure Theorem)

Let K be a field of constants, $F_0 = K(x)$, and $F_n = K(x)(\theta_1, \ldots, \theta_n)$ where θ_j is either

(i) algebraic over $F_{j-1} = F_0(\theta_1, \ldots, \theta_{j-1})$ or
(ii) $\theta_j = \log u_j$ for some $u_j \in F_{j-1}$ or
(iii) $\theta_j = e^{w_j}$ for some $w_j \in F_{j-1}$.

Then (i) $h = e^g$ where $g \notin K$ is algebraic over F_n iff $\exists\, c_i \in \mathbb{Q}$ s.t.

$$g + \sum_i c_i w_i \in K$$
$= k$ a const.

and (ii) $h = \log f$ where $f \notin K$ is algebraic over F_n iff $\exists\, k_j \in \mathbb{Z}$, $k_0 \neq 0$ s.t.

$$f^{k_0} \cdot \prod_{j \neq 0} u_j^{k_j} \in K$$
$= k$ a constant

Given $\int f(x)\,dx$, apply the theorem to construct F_n s.t. $f(x) \in F_n$, $\theta_i \notin F_{i-1}$ and transcendental (θ_i not also algebraic).

E.g. $e^{\frac{1}{2}x}$ and e^{2x} are algebraic over $\mathbb{Q}(x)(\theta_1 = e^x)$
$\log x^{-1}$ and $\log x^2$ are algebraic over $\mathbb{Q}(x)(\theta_1 = \log x)$

Example $\int \left(x \ln(2x) - \frac{1}{x}\ln(x+1) + x^2 \ln(x^2+x) \right) dx$

$F_1 = \mathbb{Q}(x)\,(\theta_1 = \ln(2x))$

$F_2 = \mathbb{Q}(x)\,(\theta_1 = \ln(\underset{u_1}{2x}),\ \theta_2 = \ln(\underset{u_2}{x+1}))$

$h = \ln(\underset{f}{x^2+x})$

Is $(x^2+x)^{k_0 = 1} \cdot (2x)^{k_1 = -1} \cdot (x+1)^{k_2 = -1} = k = \frac{1}{2}$?

$\downarrow$

Take ln $1 \cdot \ln(x^2+x) - 1 \cdot \ln(2x) - 1 \cdot \ln(x+1) = \ln\left(\frac{1}{2}\right)$

$\Longrightarrow \ln(x^2+x) = \ln(2x) + \ln(x+1) - \ln 2.$

$$\int f\,dx = \int \left[x \ln(2x) - \frac{1}{x}\ln(x+1) + x^2 \ln(x^2+x) \right] dx$$
$$= \int \left[x\theta_1 - \frac{1}{x}\theta_2 \overset{f}{+} x^2(\theta_1 + \theta_2 - \ln 2) \right] dx$$

$f \notin F_2 = \mathbb{Q}(x)\,(\theta_1 = \ln(2x),\ \theta_2 = \ln(x+1))$

$f \in F_2 = \underbrace{\mathbb{Q}(\ln 2)}_{K}(x)\,(\theta_1 = \ln(2x),\ \theta_2 = \ln(x+1)).$

So in applying Th 12.1 we may need to add a constant to K.

Example. $\int \frac{e^{2x-3}}{1+e^{x+1}} \log x \, dx$

$F_1 = \mathbb{Q}(x)\,(\theta_1 = \log x)$

$F_2 = \mathbb{Q}(x)\,(\theta_1 = \log \underset{u_1}{x},\ \theta_2 = e^{\underset{w_2}{x+1}})$

$h = e^{\underline{2x-3} = g}$.

Is h algebraic over F_2?

Is $g + c_2 \cdot w_2 = k$ a constant?

Try to Solve $g' + c_2 w_2' = 0$ for $c_2 \in \mathbb{Q}$

Have $(2x-3) + c_2(x+1) = k$

Solve $2 + c_2 \cdot 1 = 0 \Rightarrow c_2 = -2$.

$k = (2x-3) - 2(x+1) = \underline{\underline{-5}}$

So $h = e^{2x-3}$ is algebraic over F_2.

Have $g + c_2 \cdot w_2 = k$ ✓ $\quad (2x-3) - 2(x+1) = -5$

$\Rightarrow e^{g + c_2 w_2} = e^k \quad \Rightarrow e^{(2x-3)-2(x+1)} = e^{-5}$

$\Rightarrow e^g = (e^{w_2})^{-c_2} \cdot e^k \quad \Rightarrow e^{2x-3} = (e^{x+1})^2 \cdot e^{-5}$.

$$\int \frac{e^{2x-3}}{1+e^x} \log x \, dx = \int \frac{e^{-5}(e^{(x+1)})^2}{1+e^{x+1}} \log x \, dx = \int \frac{e^{-5}\theta_2^2 \; \theta_1}{1+\theta_2}.$$

$F_2 = \underbrace{\mathbb{Q}(e)}_{K}(x)(\theta_1 = \log x, \; \theta_2 = e^{x+1})$.