Assignment 5, MACM 204, Fall 2012

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Due Tuesday November 26th at 4:30pm at the beginning of class.
Late penalty: -20% for up to 48 hours late. 0 after that.
Bonus: If you were previously assigned a late penalty and you hand in this assignment 24 hours before it is due I will credit you back for the marks you lost on that assignment.
There are 8 questions. Attempt all questions.

▼ Question 1

Consider a chemical reaction involving chemicals A, B, and C in which A is converted to B at a rate k_1 and chemical B is converted to C at a rate of k_2 as illustrated in the compartment model below. I created the figure in Maple using the "Canvas" option under the insert menu, which I didn't know about but is very useful. Letting $A(t), B(t), C(t)$ be the amount of chemical A, B, C at time t we can model the chemical reactions with the differential equations

$$A'(t) = -k_1 \cdot A(t),$$
$$B'(t) = k_1 \cdot A(t) - k_2 \cdot B(t),$$
$$C'(t) = k_2 \cdot B(t).$$

Part (a): Solve the differential equations using dsolve for the initial value $A(0) = N, B(0) = 0, C(0) = 0$. Here $N > 0$ is the initial amount of chemical A. Try to simplify the solutions as best you can.

You will see that there is a factor of $k_1 - k_2$ in the denominators which means the solutions are not valid for the case $k_1 = k_2$. So solve the differential equations for this case too. Again, try to simplify the solutions.

Part (b): Now, looking at the model, can you predict what $A(\infty), B(\infty), C(\infty)$ are? You don't need to do any math to figure this out. For the case $k_1 = k_2$ try also taking the limit of $A(t), B(t), C(t)$ as $t \to \infty$. You will need to tell Maple that $k_2 > 0$. You can do this using the assuming statement like this

`> limit( A(t), t = infinity ) assuming k2>0;`

Part(c): For $N = 5, k_1 = 0.2, k_2 = 0.1$, solve the differential equations and graph $A(t), B(t), C(t)$ verses t on the same graph for a suitable domain using the plot command. Attach a legend = ["A","B","C"]. To do this you need to convert from a set of functions to a list of functions.
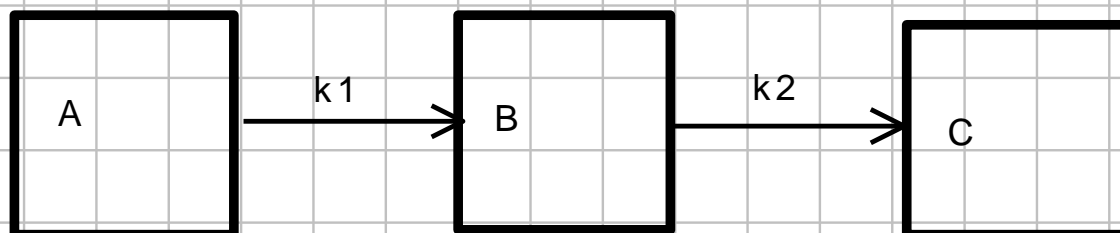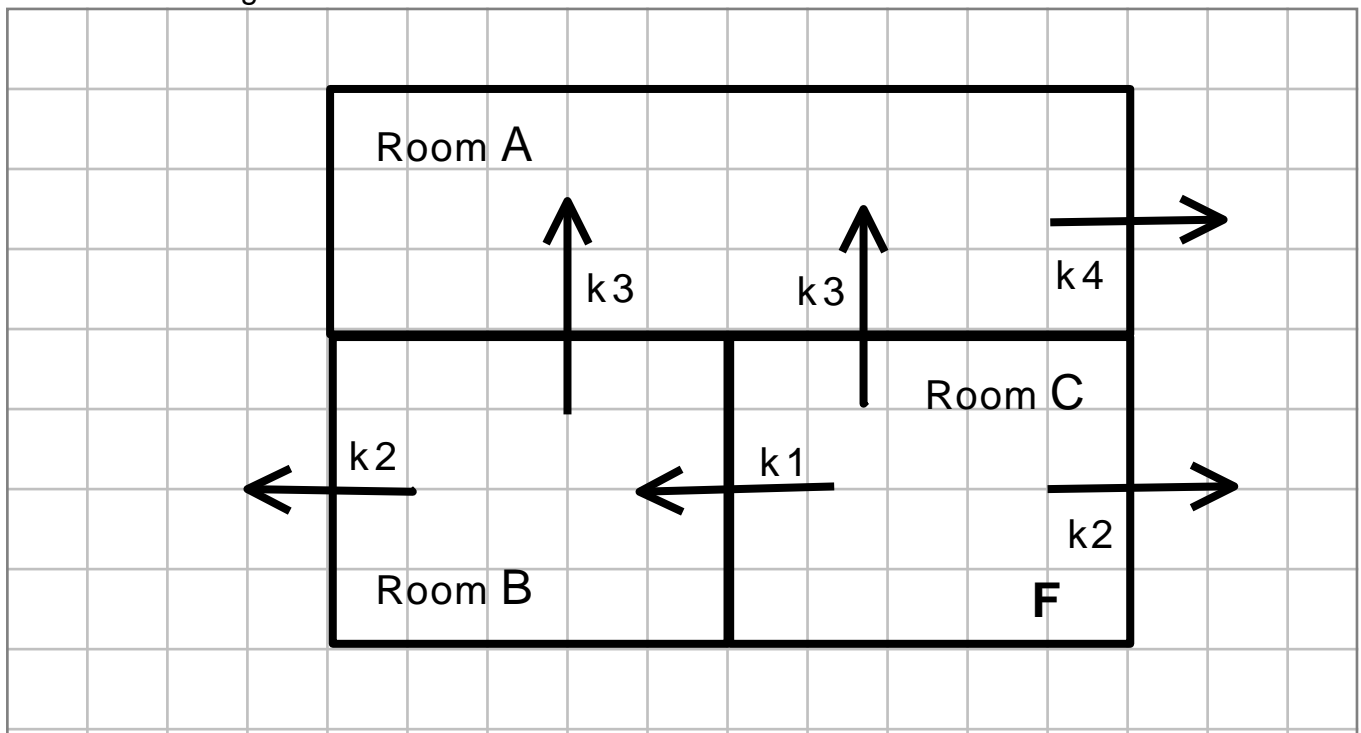
► Solution 1

▼ Question 2

Shown in the figure below is a house with three rooms, A, B and C. Rooms B and C are the same size and shape. There is a furnace **F** in room C which heats room C. Let $A(t), B(t), C(t)$ be the temperature at time t in rooms A, B, C respectively and let Am be the outside temperature. Shown in the figure are the cooling rate constants k_1, k_2, k_3, , k_4 for how heat moves through the walls of the house.

Part (a) Write down a system of 3 differential equations $A'(t) = ...,$ $B'(t) = ...,$ $C'(t) = ...$ for the house.

Part (b) Determine the temperature equilibrium point as a function of $F, Am, k1$, $k2, k3, k4$. Try to simplify the formulas by writing them in the form Am $+ f(k1, k2, k3, k4) \cdot F$.

Part (c) For $F = 5$, $Am = 0$, $k1 = 0.3$, $k2 = 0.1$, $k3 = 0.3$, $k4 = 0.2$ determine the temperature equilibrium, solve the differential equations using dsolve, and compute the limit of $A(t), B(t), C(t)$ as $t \to \infty$.

Part (d) Finally, graph $A(t), B(t), C(t)$ on the same graph on a suitable domain. Note, if the graph looks wrong this could be because your differential equations are wrong. Watch out for sign errors.

► Solution 2

▼ Question 3

Shown in the figure below is lake Erie and lake Ontario and the main rivers flowing through them (the arrows). Google says that the volume of lake Erie is about $500 \cdot km^3$ and lake Ontario is about $1500 \cdot km^3$ and the amount of water flowing through the lakes is about $60 \cdot km^3$ per year. Yes, that's kilometers cubed. The goal is to model the amount

of pollution in the two lakes at time t (years). We will assume that initially, there is no pollution in either lake and that the river flowing into lake Erie is polluted and is bringing in 30 tons of pollutant per year.

Let $Er(t)$ be the amount of pollutant (in tons) in lake Erie at time t and let $On(t)$ be the amount of pollutant (in tons) in lake Ontario at time t (years).

Set up two differential equations, one for the amount of pollution in lake Erie at time t and the other for the amount of pollution in lake Ontario at time t. This problem is very much like the tank problem in the last assignment, except that here we have two tanks (two lakes).

Solve the differential equations and plot the solutions for a suitable time domain. You should see that the amount of pollutant in each lake increases from 0 to a maximum. What are the maximums?

Using the DEplot command in the DEtools package, generate a field plot with solution curves for initial values $Er(0) = 0$, $On(0) = 0$ and $Er(0) = 500$, $On(0) = 0$ and $Er(0) = 500$, $On(0) = 1000$ on the same plot.
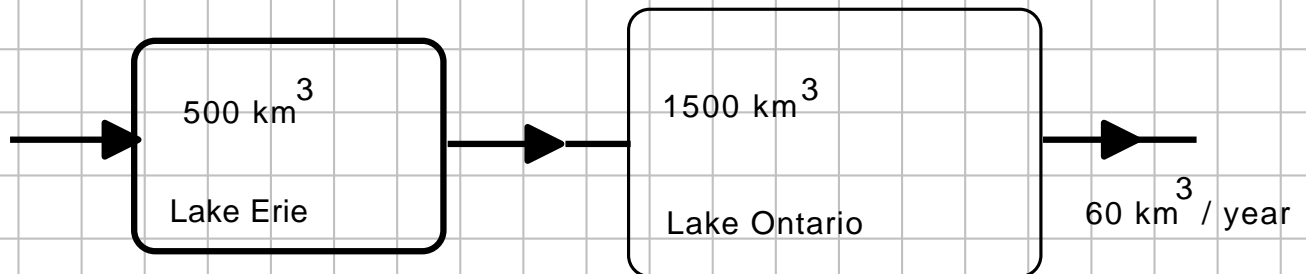
► Solution 3

▼ Question 4

The Kermack-McKendrick virus spread model (where we partition the individuals in a population into those which are susceptible, infected, and recovered), is given by

$S'(t) = -\beta \cdot S(t) \cdot I(t), \ I'(t) = \beta \cdot S(t) \cdot I(t) - \alpha \cdot I(t), \ R'(t) = \alpha \cdot I(t)$

where $S(t), I(t), R(t)$ is the proportion of the population which is susceptible, infected and recovered at time t. We cannot solve the differential equations for nice formulas but we can solve then numerically for a given set of initial values.

For $\beta = 0.3$, $\alpha = 0.1$, $S(0) = 0.99$, $I(0) = 0.01$, $R(0) = 0.00$ solve the differential equations using dsolve(..., numeric) and plot the solutions for $S(t), I(t), R(t)$ on the same plot using the odeplot command in the plots package. You should see an epidemic, that is, $I(t)$ increases to a maximum and then drops down to 0. Please include a decent legend on the plot. Looking at the plot, what proportion of the population never becomes infected, i.e., what is $S(\infty)$? [Rough estimate from the plot is fine.]

Four years ago the H1N1 flu virus hit British Columbia. Perhaps you got vaccinated? I did. Dr. Perry Kendall was the provincial health officer then. He was advising us that we

should all get a vaccination, not just to protect us individually, but to prevent H1N1 from becoming an epidemic. How can that work? The idea is that if a high enough proportion of the population gets the vaccination then they cannot contract the virus, so the virus cannot spread as easily through the population, and this can prevent an epidemic (outbreak). We can test this hypothesis in the SIR virus model. Suppose 50% of the population gets the H1N1 vaccine and 1% has the H1N1 virus (so 49% is susceptible). We can simulate this in the model using

$$S(0) = 0.49, I(0) = 0.01, R(0) = 0.50.$$

Solve the differential equations using this set of initial values to determine if this makes the virus endemic (die out) or not. If not, find, roughly, the %age of the population that must be vaccinated (this is for α=0.1, β=0.3) to make the virus endemic.

► Solution 4

The following is for questions 5 and 6.

Let $M(t)$ be the amount owed on a 30 year mortgage of $200,000 at time t years.
Suppose the annual interest rate on the mortgage is r=4%.
Suppose the term of the mortgage is 30 years, i.e., M(30) should be 0.
Suppose we pay P per year. We can use the following differential equation model interest charges and our payments.

$$M'(t) = r \cdot M(t) - P$$

If we assume the interest is paid continuously (banks usually charge interest daily which is approximately continuous) and if we assume we make the payments continuously (banks usually require us to pay monthly or weekly which is approximately continuous over 30 years) then we can estimate the annual payment as follows.

```
> de := diff( M(t),t) = r*M(t)-P;
  r := 0.04;
```

$$de := \frac{\mathrm{d}}{\mathrm{d}t} M(t) = r M(t) - P$$

$$r := 0.04$$

```
> sol := dsolve( {de, M(0)=200000}, M(t) );
```

$$sol := M(t) = 25 P + e^{\frac{1}{25} t} (200000 - 25 P)$$

To determine P we solve $M(30) = 0$ for P.

```
> eq := eval( rhs(sol), t=30.0 ) = 0;
```

$$eq := -58.00292308 P + 6.640233846 \, 10^5 = 0$$

```
> yearlypayment := solve( eq, P );
```

$$yearlypayment := 11448.10208$$

So we pay $11,448.10 dollars per year.
So if the bank requires us to make equal monthly payments (at the end of each month), we pay

```
> monthlypayment := yearlypayment/12;
```

$$monthlypayment := 954.0085067$$

per month.
Also, the total interest we pay is

$$\$143,443.06$$

▼ Question 5

Suppose you want to pay off the mortgage faster to avoid paying all that interest!!
Suppose you decide to pay P = $15,000 pear year instead.
Solve the differential equation with r = 4% per year and P = 15,000 per year and determine (i) how long it will take you to pay off the mortgage and (ii) how much interest you will pay.
Now graph (**on the same plot**) *M(t)* for *P*= $11,448.10 **and** *M(t)* with *P*= $15,000 on a suitable domain. Include a legend and title.

► Solution 5

▼ Question 6

What I want you to do is investigate how accurate the model is by starting with $M(0) = 200,000$ and simulating the daily interest charges and monthly payments over the 30 years using $P=\$11448.10$ per year. Start with M = 200,000. Add $\dfrac{r}{365} \cdot M$ to M daily and then subtract $\dfrac{P}{12}$ from M **at the end** of each month. If the model is accurate we should end up with M close to 0. Try this. You should find that you still owe a non-trivial amount. Note, months have different lengths so you need to figure out on which day of the year each month ends. For simplicity, assume there are no leap years so February always has 28 days.

Now repeat this calculation assuming that we make the payments on the 15th day of each month.
You should find that M is much closer to 0 after 30 years. Explain briefly why.
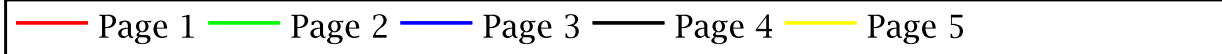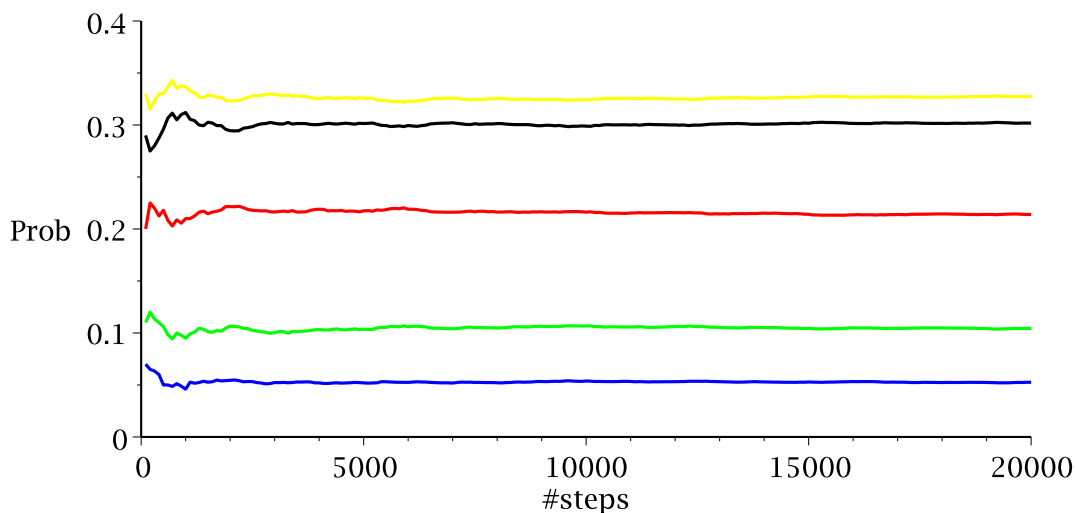
► Solution 6

▼ Question 7
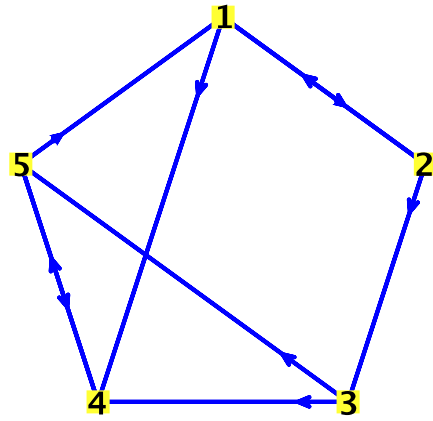
Shown above is a directed graph with 5 vertices. The graph represents 5 web pages and the directed edges represent the hyperlinks between the web pages. Input and draw the graph in Maple (see ?GraphTheory and use the ?Graph and ?DrawGraph commands).

Now, starting at vertex 1, simulate a random walk of length $N = 20,000$ steps through the graph. Store the walk in an array W. So since vertex 1 is connected to vertex 2 and 4, the random walk will go to vertex 2 or 4 with probability 0.5. So if $W_1 = 4$ this means after the first random step you are on page (vertex) 4. At the end of the random walk, for each vertex, calculate the probability the walker visits that vertex.

Now modify your code to do the following. After each 100 steps, for each vertex, calculate the probability that the walker was on that vertex. For each vertex, this generates a sequence of 200 probabilities that should converge.

Graph these 5 sequences on a single graph. Include a legend. You should get a picture like the following (Page 1 corresponds to vertex 1)

► Solution 7

▼ Question 8

The Koch snowflake (look for a picture using google) can be constructed as follows. Start with an equilateral triangle ABC. For each edge in the triangle (shown below as n= 0 is one of the edges) we will build a fractal as follows: Replace the edge by the four edges shown in the figure for n=1 as follows: divide the edge into three equal subintervals and form a small equilateral triangle from the middle edge but omit the base (the dotted line). Repeat this procedure *n* times for each new edge. The result for n=2 is shown below. Write a recursive Maple program to do this.

$$n=0 \qquad n=1 \qquad n=2$$
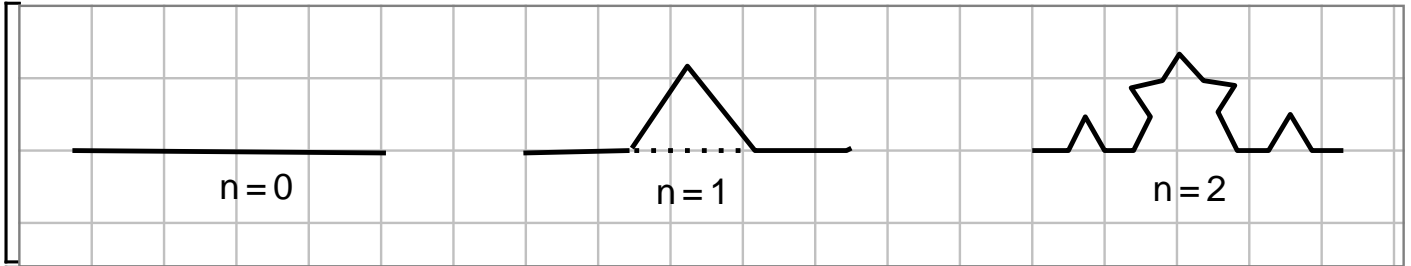
For this question, depending on your you solve it, you may need to rotate a vector by 60 degrees ($\pi/3$ radians). If you need it, the rotation matrix for rotating anti–clockwise by θ degrees/radians is

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

► Solution 8