Assignment 4, MACM 204, Fall 2013

Due Thursday November 7th at the beginning of the lab.
Late penalty: -20% for up to 72 hours late. 0 after that.

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Please attempt each question in a seperate worksheet (so that you don't destroy your previous work). Print your Maple worksheets (you may print double sided if you wish) and hand them in to me.

As usual, there are 8 questions.

Question 1

Input the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and the vectors $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in Maple.

Part (a) In Maple, calculate $2 \cdot u + w$, A^3 and A^{-1}.

Part (b) Using the commands in the **LinearAlgebra** package calculate the determinant of A, the characteristic polynomial of A in the variable x and the solutions of the linear systems $A \cdot x = u$ and $A \cdot x = w$. Note, to see all the commands in the package do

```
> ?LinearAlgebra
```

Part (c) Starting with the vector $v_1 = [1, 0]$ compute the vectors $v_2 = A \cdot v_1$, $v_3 = A \cdot v_2$, $v_4 = A \cdot v_3$, ..., $v_{10} = A \cdot v_9$

What numbers appear in the sequence of vectors $v_1, v_2, v_3, \ldots, v_{10}$?

Solution 1

Question 2

Consider again the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Part (a) Using the Eigenvalues and Eigenvectors commands in the LinearAlgebra package calculate the eigenvalues and eigenvectors of the matrix . The answers will be exact. Use the evalf command to get decimal approximations of them. Note, by default, the Eigenvectors command outputs the eigenvectors as columns in a matrix.

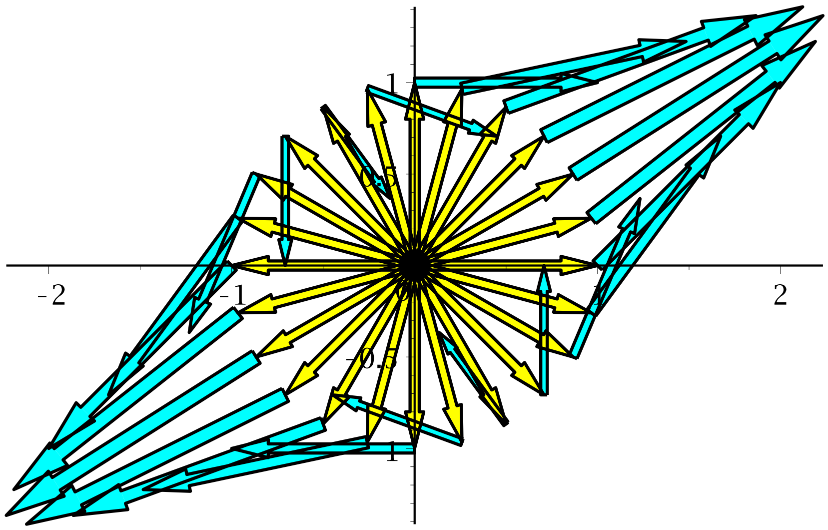
Part (b) Now, pick any unit vector e.g. $u = [1, 0]$. Compute the sequence of vectors $A \cdot u$, $A^2 \cdot u$, ..., $A^{10} \cdot u$ in a loop. After each multiplication of $v := A \cdot u$ make the vector v a unit vector, i.e., set $u := \frac{v}{\sqrt{v_1^2 + v_2^2}}$. Use decimal arithmetic. Repeat this using a different starting vector u.

Which eigenvector does $A^{10} \cdot u$ converge to?

You should find that it always converges to the eigenvector with the positive eigenvalue.

Below is a picture showing 24 unit vectors and 24 vectors obtained by multiplying A by the unit vectors.

```
> display( [seq( unitarrow[i], i=1..n )],
          [seq( imagearrow[i], i=1..n )], scaling=constrained );
```



Challenge question (**bonus marks**):
Can you explain why the sequence $A \cdot u, A^2 \cdot u, A^3 \cdot u, A^4 \cdot u, \ldots$ always converges to the eigenvector with positive eigenvalue?
Can you see how to modify the sequence so that starting with any unit vector u you can converge to the other eigenvector with a negative eigenvalue?

Solution 2

Question 3

This question related to Newton's law of cooling.
Let $T(t)$ be the temperature of a body of liquid at time t. Let $Troom$ be the room (ambient) temperature of the surrounding medium (air). The DE is
$T'(t) = k \cdot (Troom - T(t))$ where k is the cooling rate constant.
Solve the differential equation in Maple for Troom = 20 degrees and an initial temperature of 40 degrees.

Given also that $T(20) = 30$, determine k. Now compute T(60). Do all the calculations in Maple.
Finally graph $T(t)$ for $0 \le t \le 60$ together with the room temperature on a suitable domain/range .

Solution 3

Question 4

Carbon 14 decays into Nitrogen 14. Using Google, find the half life *H* of Carbon 14.
The differential equation modeling radioactive decay is

$$y'(t) = -k \cdot y(t)$$

where *k* is the decay constant and y(0) is the initial concentration of Carbon 14. Given the half life is *H*, that is, given that $y(H) = \frac{y(0)}{2}$, determine k. You can do this one by hand at first but then do it in Maple.
Solve the DE in Maple and graph the solution for $y(0) = 1$ on a suitable domain.

Solution 4

Question 5

Suppose we have a 400 liter tank. Suppose 8 litres per minute of salt water (brine) flows into the tank at the top and then flows out of the tank at the bottom. Assume for simplicity that the salt water in the tank is stirred so that its concentration is uniform in the tank. Let $S(t)$ be the amount of salt, in grams, in the tank at time t minutes.

Suppose the salt water flowing into the tank has concentration 100 grams per liter.

Find the differential equation to model the change in S(t).

Assuming there is no salt in the tank at time t=0 solve the differential equation using Maple.

What is S(∞)? That is, how much salt is in the tank after a long time?

Now graph S(t) for a suitable domain.

Solution 5

Question 6

The logistic growth with harvesting model for a population $y(t)$ at time t is given by

$$y'(t) = \boldsymbol{a} \cdot y(t) \cdot \left(Y_{\max} - y(t)\right) - H$$

Here $Ymax$ is the maximum sustainable population of the environment, **a** is a constant and *H* is a constant harvesting rate. For Ymax = 8000, a=0.0001, and H = 1000, using the DEplot command, graph y(t) for $0 \le t \le 10$ for the initial values $y(0)$ in 1000, 5000, 8000 and 10000.

Now determine populations y for which y' = 0, i.e., find the initial polulations for which there is no growth or decline. You should get two. Graph these on the same graph - you should get two straight lines.

Solution 6

Question 7

Consider the following differential equation that we used to model a mortgage payment.

$y'(t) = r \cdot y(t) - D$

Solve the DE by hand using the method of substitution.
Assume y(0)=M to determine the constant C.

Solution 7

Question 8

Consider an equilateral triangle A B C.
The **Fractal game** is to do the following

Pick an initial point x_0 anywhere in the triangle.
for k from 1 to n do
Toss a die
If you get 1 or 2 set x_k to be half way from x_{k-1} to A.
If you get 3 or 4 set x_k to be half way from x_{k-1} to B.
Otherwise set x_k to be half way from x_{k-1} to C.

This generates a sequence on n points $x_1, x_2, ..., x_n$ which, not, must all be inside the triangle.

Program this in Maple and graph the points for n at least 5000. What picture do you get?

Note, to simulate a die use

```
> die := rand(1..6):
```

Then die(); will generate a random integer on [1,6]. E.g.

```
> die();
```

4

```
> die();
```

3

```
> die();
```

6

Note, you can use Maple lists $[x, y]$ to represent the points A, B, C and the x_k .

▶ Solution 8