Assignment 4, MACM 204, Fall 2012

Due Wednesday November 14th at 3:30pm at the beginning of the second lab.
Late penalty: -20% for up to 24 hours late. 0 after that.

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Please attempt each question in a seperate worksheet (so that you don't destroy your previous work). Print your Maple worksheets (you may print double sided if you wish) and hand them in to me.

There are 8 questions.

Question 1

Let $M(n, a, b)$ be an *n* by *n* matrix with *a* on the diagonal and *b* on the sub-diagonal and sup-diagonal and 0 elsewhere. For example, $M(4, 2, 1)$ is the matrix

```
> Matrix(4,4,[[2,1,0,0],[1,2,1,0],[0,1,2,1],[0,0,1,2]]);
```

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \tag{1.1}$$

Compute the det($M(n, 2, 1)$) for n=3,4,5,... in a loop. Hence find a formula for det($M(n, 2, 1)$). To compute a determinant use the Determinant command from the LinearAlgebra package.

Solution 1

Question 2

Given the following survival rates and fertility rates for a seal population.

```
> S1 := 0.626; # 4yr survival rate for seal pubs
  S2 := 0.808; # 4yr survival rate for young seal adults
  S3 := 0.808; # 4yr survival rate for mature seal adults
  F1 := 0.0;   # 4yr fertility rate for seal pubs
  F2 := 1.26;  # 4yr fertility rate for young adults
  F3 := 2.00;  # 4yr fertility rate for mature adults
```

$$S1 := 0.626$$
$$S2 := 0.808$$
$$S3 := 0.808$$
$$F1 := 0.$$
$$F2 := 1.26$$
$$F3 := 2.00 \tag{3.1}$$

Here is the Leslie age distribution matrix.

```
> L := Matrix([[F1,F2,F3],
              [S1,0,0],
             [0,S2,S3]] );
```

$$L := \begin{bmatrix} 0. & 1.26 & 2.00 \\ 0.626 & 0 & 0 \\ 0 & 0.808 & 0.808 \end{bmatrix} \quad (3.2)$$

Starting with an initial population vector $P_0 := \begin{bmatrix} 1.0 \\ 0 \\ 0 \end{bmatrix}$ (in units of 1000 female seals) run the model for 40 years (10 timesteps). Compute P_{10} the population vector after 40 years AND D_{10} the population **distribution** vector after 40 years. Please print P_9, P_{10}, D_9 and D_{10}.

You should find that the population distribution stabilizes after 40 years. That means P_{10} is (has converged to) an eigenvector of L.

Determine the eigenvalue from the data. Now check that your result is right by computing the eigenvalues of L using Maple. The value you computed should be right to 3 decimal places.

Solution 2

Question 3

This question related to Newton's law of cooling.
Let $T(t)$ be the temperature of a body of liquid at time t. Let $Troom$ be the room (ambient) temperature of the surrounding medium (air). The DE is $T'(t) = k \cdot (Troom - T(t))$ where k is the cooling rate constant.
Solve the differential equation in Maple for Troom = 20 degrees and an initial temperature of 40 degrees.
Given also that $T(20) = 30$, determine k. Now compute T(60). Do all the calculations in Maple.
Finally graph $T(t)$ for $0 \le t \le 60$ together with the room temperature on a suitable domain/range .

Solution 3

Question 4

Let $P(t)$ be the amount owed on a 30 year mortgage of \$200,000 at time t years.
Suppose the annual interest rate on the mortgage is *r%* where *r*=5%.
Suppose the term of the mortgage is 30 years, i.e., P(30) should be 0.
Suppose we pay \$D per year. We can use the following differential equation to compute P(t)

$$P'(t) = r \cdot P(t) - D$$

If we assume the interest is paid continuously (banks usually charge interest daily which is approximately continous) and if we assume we make the payments continously (banks usually require us to pay monthly or weekly which is approximately continous over 30 years) then we can compute the payment as follows.

```
> de := diff( P(t),t) = r*P(t)-D;
  r := 0.05;
  sol := dsolve( {de, P(0)=200000}, P(t) );
```

$$de := \frac{d}{dt} P(t) = r P(t) - D$$

$$r := 0.05$$

$$sol := P(t) = 20\,D + e^{\frac{1}{20} t} (200000 - 20\,D) \quad \textbf{(7.1)}$$

```
> eq := eval( rhs(sol), t=30.0 ) = 0;
```

$$eq := -69.63378140\,D + 8.963378140\,10^{5} = 0 \quad \textbf{(7.2)}$$

```
> yearlypayment := solve( eq, D );
```

$$yearlypayment := 12872.16917 \quad \textbf{(7.3)}$$

So we pay $12,872.17 dollars per year or 12872.17/12 = $1072.68 per month.
First, please calculate the total interest we paid to the bank.

What I want you to do is simulate the interest charges and monthly payments over the 30 years by starting with P= 200,000 and computing the interest daily and making the payments monthly for 30 years. So you add $\frac{r}{365} \cdot P$ to P daily and then subtract $1072.68 from P **at the end** of each month. If the DE is accurate we should end up with P close to 0. Try this. You should find that you still owe a non-trivial amount. Note, if you paid the $1072.68 at the beginning of each month, instead, you would more than pay off the mortgate. Note, the months have different lengths so you need to figure out which day of the year each month ends.

Now repeat this calculation assuming that we make the payments daily instead of monthly. So pay off $12872.17 / 365 = $35.266 per day. This will make the payments more continuous. You should find that P(30) is close to 0.

▶ Solution 4

▼ Question 5

Suppose we have a 400 liter tank. Suppose 8 litres per minute of salt water (brine) flows into the tank at the top and then flows out of the tank at the bottom. Assume for simplicity that the salt water in the tank is stirred so that its concentration is uniform in the tank. Let $S(t)$ be the amount of salt, in grams, in the tank at time t minutes.

Suppose the salt water flowing into the tank has concentration 100 grams per liter.

Find the differential equation to model the change in S(t).

Assuming there is no salt in the tank at time t=0 solve the differential equation using Maple.

What is S(∞)? That is, how much salt is in the tank after a long time?

Now graph S(t) for a suitable domain.

Solution 5

Question 6

The logistic growth with harvesting model for a population $y(t)$ at time t is given by

$$y'(t) = \boldsymbol{a} \cdot y(t) \cdot \left(Y_{\max} - y(t)\right) - H$$

Here $Ymax$ is the maximum sustainable population of the environment, **a** is a constant and *H* is a constant harvesting rate. For Ymax = 8000, a=0.1, and H = 1000, using the DEplot command, graph y(t) for $0 \le t \le 10$ for the initial values $y(0)$ in 1000, 5000, 8000 and 10000.

Now determine populations *y* for which *y* ' = 0, i.e., find the initial polulations for which there is no growth or decline. You should get two. Graph these on the same graph - you should get two straight lines.

Solution 6

Question 7

Consider the following differential equation that we used to model a mortgage payment.

$$y'(t) = r \cdot y(t) - \mathrm{D}$$

Solve the DE by hand using the method of substitution.
Assume y(0)=M to determine the constant C.

Solution 7

Question 8

Carbon 14 decays into Nitrogen 14. Using Google, find the half life *H* of Carbon 14.
The differential equation modeling radioactive decay is

$$y'(t) = -k \cdot y(t)$$

where *k* is the decay constant and y(0) is the initial concentration of Carbon 14. Given the half life is *H*, that is, given that $y(H) = \frac{y(0)}{2}$, determine k. You can do this one by hand at first but then do it in Maple.
Graph the solution for $y(0) = 1$ on a suitable domain.

Solution 8