

MACM 401/MATH 701/MATH 819/CMPT 881  
Assignment 4, Spring 2013.

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This assignment is to be handed in by Monday March 11th at the start of class.

For problems involving Maple calculations and Maple programming, you should submit a printout of a Maple worksheet of your Maple session.

Late Penalty:  $-20\%$  for up to 24 hours late. Zero after that.

**Question 1:  $p$ -adic Lifting (15 marks)**

Reference: Section 6.3.

- (a) By hand, determine the  $p$ -adic representation of the integer  $u = 116$  for  $p = 5$ , first using the positive representation, then using the symmetric representation for  $\mathbb{Z}_p$ .

Using Maple, determine the  $p$ -adic representation for the polynomial

$$u(x) = 28x^2 + 24x + 58$$

with  $p = 3$  using, first using the positive representation for  $\mathbf{Z}_3$ , then the symmetric representation.

- (b) Determine the cube-root, *if it exists*, of the following polynomials

$$a(x) = x^6 - 531x^5 + 94137x^4 - 5598333x^3 + 4706850x^2 - 1327500x + 125000,$$

$$b(x) = x^6 - 406x^5 + 94262x^4 - 5598208x^3 + 4706975x^2 - 1327375x + 125125$$

using reduction mod 5 and linear  $p$ -adic lifting. You will need to derive the update formula by modifying the update formula for computing the  $\sqrt{a(x)}$ .

Factor the polynomials so you know what the answers are. Express the answer in the  $p$ -adic representation. To calculate the initial solution  $u_0 = \sqrt[3]{a} \pmod{5}$  use any method. Use Maple to do all the calculations.

**Question 2: Hensel lifting (15 marks)**

Reference: Section 6.4 and 6.5.

- (a) Given

$$a(x) = x^4 - 2x^3 - 233x^2 - 214x + 85$$

and image polynomials

$$u_0(x) = x^2 - 3x - 2 \quad \text{and} \quad w_0(x) = x^2 + x + 3,$$

satisfying  $a \equiv u_0 w_0 \pmod{7}$ , lift the image polynomials using Hensel lifting to find (if there exist)  $u$  and  $w$  in  $\mathbb{Z}[x]$  such that  $a = uw$ .

(b) Given

$$b(x) = 48x^4 - 22x^3 + 47x^2 + 144$$

and an image polynomials

$$u_0(x) = x^2 + 4x + 2 \quad \text{and} \quad w_0 = x^2 + 4x + 5$$

satisfying  $b \equiv 6u_0w_0 \pmod{7}$ , lift the image polynomials using Hensel lifting to find (if there exist)  $u$  and  $w$  in  $\mathbb{Z}[x]$  such that  $b = uw$ .

### Question 3: Determinants (20 marks)

Consider the following 3 by 3 matrix  $A$  of polynomials in  $\mathbb{Z}[x]$  and its determinant  $d$ .

```
> P := () -> randpoly(x, degree=2, dense):
```

```
> A := Matrix(3,3,P);
```

$$A := \begin{bmatrix} -55 - 7x^2 + 22x & -56 - 94x^2 + 87x & 97 - 62x \\ -83 - 73x^2 - 4x & -82 - 10x^2 + 62x & 71 + 80x^2 - 44x \\ -10 - 17x^2 - 75x & 42 - 7x^2 - 40x & 75 - 50x^2 + 23x \end{bmatrix}$$

```
> d := LinearAlgebra[Determinant](A);
```

$$d := -224262 - 455486x^2 + 55203x - 539985x^4 + 937816x^3 + 463520x^6 - 75964x^5$$

- (a) Let  $A$  be an  $n$  by  $n$  matrix of polynomials in  $\mathbb{Z}[x]$  and let  $d = \det(A)$ . Develop a modular algorithm for computing  $d = \det(A) \in \mathbb{Z}[x]$ . Your algorithm will compute determinants of  $A$  modulo a sequence of primes and apply the CRT. For each prime  $p$  it will compute the determinant in  $\mathbb{Z}_p[x]$  by evaluation and interpolation. In this way we reduce computation of a determinant of a matrix over  $\mathbb{Z}[x]$  to many computations of determinants of matrices over  $\mathbb{Z}_p$ , a field, for which ordinary Gaussian elimination, which does  $O(n^3)$  arithmetic operations in  $\mathbb{Z}_p$ , may be used.

You will need bounds for  $\deg d$  and  $\|d\|_\infty$ . Use primes  $p = [101, 103, 107, \dots]$  and use Maple to do Chinese remaindering. Use  $x = 1, 2, 3, \dots$  for the evaluation points and use Maple for interpolation. Implement your algorithm in Maple and test it on the above example.

To reduce the coefficients of the polynomials in  $A$  modulo  $p = 7$  in Maple use

```
> B := A mod p;
```

To evaluate the polynomials in  $B$  at  $x = \alpha$  modulo  $p$  in Maple use

```
> C := eval(B, x=alpha) mod p;
```

To compute the determinant of a matrix  $C$  over  $\mathbb{Z}_p$  in Maple use

```
> Det(C) mod p;
```

- (b) Suppose  $A$  is an  $n$  by  $n$  matrix over  $\mathbb{Z}[x]$  and  $A_{i,j} = \sum_{k=0}^d a_{i,j,k}x^k$  and  $|a_{i,j,k}| < B^m$ . That is  $A$  is an  $n$  by  $n$  matrix of polynomials of degree at most  $d$  with coefficients at most  $m$  base  $B$  digits long. Assume the primes satisfy  $B < p < 2B$  and that arithmetic in  $\mathbb{Z}_p$  costs  $O(1)$ . Estimate the time complexity of your algorithm in big  $O$  notation as a function of  $n$ ,  $m$  and  $d$ . Make reasonable simplifying assumptions such as  $n < B$  and  $d < B$  as necessary. Also helpful is  $\ln n! < n \ln n$ . State your assumptions.

#### Question 4: Factorization in $\mathbb{Z}[x]$ (30 marks)

Factor the following polynomials in  $\mathbb{Z}[x]$ .

$$p_1 = x^{10} - 6x^4 + 3x^2 + 13$$

$$p_2 = 8x^7 + 12x^6 + 22x^5 + 25x^4 + 84x^3 + 110x^2 + 54x + 9$$

$$p_3 = 9x^7 + 6x^6 - 12x^5 + 14x^4 + 15x^3 + 2x^2 - 3x + 14$$

$$p_4 = x^{11} + 2x^{10} + 3x^9 - 10x^8 - x^7 - 2x^6 + 16x^4 + 26x^3 + 4x^2 + 51x - 170$$

For each polynomial, first compute its square free factorization. Use the Maple command `gcd(...)` to do this. Now factor each non-linear square-free factor as follows. Use the Maple command `Factor(...)` mod `p` to factor the square-free factors over  $\mathbb{Z}_p$  modulo the primes  $p = 13, 17, 19, 23$ . From this information, determine whether each polynomial is irreducible over  $\mathbb{Z}$  or not. If not irreducible, try to discover what the irreducible factors are by considering combinations of the modular factors and Chinese remaindering (if necessary) and trial division over  $\mathbb{Z}$ .

Using Chinese remaindering here is not efficient in general. Why? Thus for the polynomial  $p_4$ , use Hensel lifting instead. That is, using a suitable prime of your choice from 13, 17, 19, 23, Hensel lift each factor mod  $p$ , then determine the irreducible factorization of  $p_4$  over  $\mathbb{Z}$ .

#### Question 5: (10 marks) (MATH 701 and MATH 819 students only)

For the Sparse Multivariate Polynomial data structure that you designed and implemented on assignment 2, implement a Maple procedure `SMPdiv(A,B)` that outputs the quotient  $Q$  if  $B|A$  otherwise outputs FAIL. Test your routine on the examples in Question 6 below.

#### Question 6: (20 marks) (CMPT 881 students only)

If you used a recursive form for the SMP polynomial data structure on your last assignment, use a distributed form this time. And if you used a distributed form on your last assignment use a recursive form this time. Implement the same 5 Maple procedures

- `Maple2SMP` - to convert from Maple's expanded form to SMP,
- `SMP2Maple` - to convert from SMP to Maple's expanded form,
- `SMPadd` - to add two polynomials,
- `SMPmul` - to multiply two SMP polynomials,
- `SMPdiv` - to divide two SMP polynomials.

Use Maple to do coefficient and exponent arithmetic. Test your routine on the following

```
> a := randpoly([x,y,z],degree=6,terms=15);
> b := randpoly([x,y,z],degree=6,terms=15);
> A := Maple2SMP(a);
> B := Maple2SMP(b);
> C := SMPadd(A,B);
> a+b - SMP2Maple(C));
> C := SMPmul(A,B);
> expand(a*b) - SMP2Maple(C); # should output 0
> SMPdiv(A,B); # should output FAIL
> Q := SMPdiv(C,A);
> expand(b-SMP2Maple(Q)); # should output 0
```