Resultants

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Let $a, b \in K[x]$, K a field e.g. $\mathbb{Q}$.

Let $a = \sum_{i=0}^{m} a_i x^i = a_m \prod_{i=1}^{m} (x - \alpha_i)$. $b = \sum_{i=0}^{n} b_i x^i = b_n \prod_{i=1}^{n} (x - \beta_i)$.

Def. $res(a, b, x) = a_m^n \cdot b_n^m \prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j)$.

Th. 1 $\deg(\gcd(a,b)) > 0 \iff res(a,b,x) = 0$

Th 2. $res(a,b,x) = a_m^n \prod_{j=1}^{m} \underbrace{b_n \prod_{i=1}^{n} (\alpha_i - \beta_j)}_{b(\alpha_i)} = a_m^n \prod_{j=1}^{m} b(\alpha_i)$.

Th 3. $res(a, bc, x) = res(a,b,x) \cdot res(a,c,x)$. Proof. Ex.

Ex. $res(\underbrace{x^2-1}_{a}, \underbrace{x^2-3}_{b}, x) = 1^2 \cdot 1^2 \cdot (1-\sqrt{3})(1+\sqrt{3})(-1-\sqrt{3})(-1+\sqrt{3})$

$= (1-3)(1-3) = +4 \in \mathbb{Q}$.

$\alpha_1 = 1$ $\beta_1 = +\sqrt{3}$

$\alpha_2 = -1$ $\beta_2 = -\sqrt{3}$

Th 2. $res(\overset{a}{x^2-1}, \overset{b}{x^2-3}, x) = 1^2 b(1) \cdot b(-1)$.

$\alpha_1 = 1$ $= (1-3) \cdot (1-3) = 4$.

$\alpha_2 = -1$

Th 4. $res(a,b,x) \in K$.

Ex. $a, b \in \underbrace{\mathbb{Q}(x)}_{=K}[z]$.

$a = z^2 - 2$. $R = res(a,b,z) = 1^2 \cdot b(\sqrt{2})\, b(-\sqrt{2})$

$b = (x - 2z)^2 - 2$. $= ((x - 2\sqrt{2})^2 - 2)((x + 2\sqrt{2})^2 - 2)$

$\alpha_1 = \sqrt{2}$ $= x^4 - 20x^2 + 36 \in \mathbb{Q}(x)$.

$\alpha_2 = -\sqrt{2}$.

Note. $\deg(R, x) = \deg(b, x) \cdot \deg(a, z)$.

Th... Let $a = \sum^{m} a_i x^i$ and $b = \sum^{n} b_i x^i$. Then

Thm. Let $a = \sum_{i=0}^{m} a_i x^i$ and $b = \sum_{i=0}^{n} b_i x^i$. Then

$$\text{res}(a,b,x) = \det\left(\begin{bmatrix} a_m & & 0 & b_n & & 0 \\ \vdots & a_m & & \vdots & b_n & \\ a_0 & \vdots & a_m & b_0 & \vdots & b_n \\ & a_0 & \vdots & & b_0 & \vdots \\ 0 & & a_0 & 0 & & b_0 \end{bmatrix}\right) \leftarrow \text{Sylvester's matrix.}$$

(n times for the a columns, m times for the b columns)

$$\text{res}(x^2-1, x^2-3, x) = \det\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -3 & 0 \\ 0 & -1 & 0 & -3 \end{bmatrix}\right) \underset{R_4 \leftarrow R_4 + R_2}{\overset{R_3 \leftarrow R_3 + R_1}{\sim}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = +4.$$

It follows that if $a, b \in R[x]$ then $\text{res}(a,b,x) \in R$.

What's the best way to compute $\text{res}(a,b,x)$?

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> resultant(a,b,x);
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$$\text{res}(a,b) = a_m^n b_n^m \prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j)$$

① $\text{res}(b,a) = (-1)^{nm} \text{res}(a,b)$

② $\text{res}(c \in R, b) = c^n \cdot b_m^0 \cdot 1 = c^n$

Suppose $m \geq n > 0$ (a, b). Let $a = bq + r$ with $r = 0$ or $\deg r < \deg b$. $\ell = \deg(r)$.

CASE $r = 0$: $\text{res}(a,b) = \text{res}(bq, b) = 0$ by Th 1.

CASE $r \neq 0$: $\text{res}(a,b) = (-1)^{mn} \text{res}(b,a)$

$b(\beta_j) = 0.$

$$\overset{\text{Th 2}}{=} (-1)^{nm} b_n^m \prod_{j=1}^{n} (a(\beta_j))$$

$$= (-1)^{mn} b_n^m \prod_{j=1}^{n} (b(\beta_j) \cdot q(\beta_j) + r(\beta_j)), \quad b(\beta_j) = 0$$

$$\text{res}(a,b) = (-1)^{mn} b_n^m \prod_{j=1}^{n} r(\beta_j).$$

$$\text{res}(a,b) = (-1)^{mn} b_n^m \prod_{j=1}^{n} r(\beta_j).$$

$$\text{res}(r,b) = (-1)^{\ell n} \text{res}(b,r) \overset{\text{Thm}}{=} (-1)^{\ell n} \cdot b_n^{\ell} \prod_{j=1}^{n} r(\beta_j).$$

$$\frac{\text{res}(a,b)}{\text{res}(r,b)} = (-1)^{mn-\ell n} \cdot b_n^{m-\ell} \cdot 1.$$

(3) $$\text{res}(a,b) = (-1)^{mn-\ell n} \cdot b_n^{m-\ell} \cdot \text{res}(r,b).$$

We can compute $\text{res}(a,b)$ using the Euc. Alg. with $O(nm)$ mults. in R a field.