Rational Expression Simplification with Algebraic Side Relations

Roman Pearce   Michael Monagan

Introduction

Our problem was to simplify rational expressions in the presence of algebraic side relations. This problem can occur quite easily, such as when an expression involves functions. For example:

$$\frac{(1+\cos(x))\sin(x)-\cos^2(x)+1}{\cos^4(x)-2\cos^2(x)+\sin(x)+1} \longrightarrow \frac{\sin(x)+\cos(x)+1}{(1-\cos^2(x))\sin(x)+1}$$

The difficulty of this problem is threefold:

- A single expression can have many equivalent representations.
- Expressions may not factor uniquely, e.g.

$$\sin^2(x) = (1+\cos(x))\,(1-\cos(x))$$

- Sometimes we must simplify in a way that does not correspond to cancelling a common divisor. i.e. $\frac{a}{b} \to \frac{c}{d}$ but $c \nmid a$ and $d \nmid b$.

Our Idea

Rather than work with the expressions themselves, we had to construct *classes* of equivalent expressions and compute with them.

- We first replace any functions by variables. The result is a rational expression $\frac{a}{b}$ where a and b are in a polynomial ring $k[x_1,\ldots,x_n]$.
- The side relations are written in the form $f(x_1,\ldots,x_n)=0$.
- The set of all expressions equivalent to zero is an *ideal* of $k[x_1,\ldots,x_n]$.

Definition Let R be a ring, a set $I \subseteq R$ is an ideal if

i) $f+g \in I$ for all $f,g \in I$

ii) $f\,h \in I$ for all $f \in I$ and all $h \in R$

We let $I = \langle f_1,\ldots,f_s\rangle$ be the ideal generated by our side relations, then we have the following correspondence:

Elements of $k[x_1,\ldots,x_n]\,/\,I$		Ideals of $k[x_1,\ldots,x_n]$
f	$\longleftrightarrow$	$\langle f, I\rangle$

So that instead of computing with a polynomial f, we will compute with the ideal generated by f together with all of the side relations.

Making it Work

- To compute with ideals of $k[x_1,\ldots,x_n]$ we use *Gröbner bases*.
- Simply put, a Gröbner basis contains all of the minimal elements of an ideal. We can use them to cancel out the parts of an expression which are in the ideal, and simplify what is left to a canonical form.
- Perhaps most importantly, they allow us to divide by an ideal's generators in much the same way as with ordinary univariate polynomials.

Example The polynomial y^2+1 is in the ideal $\langle xy+1, x^2+1\rangle$. Using Gröbner bases we can confirm this fact, and also obtain

$$y^2+1 = (1-xy)\,(xy+1) + y^2\,(x^2+1)$$

We can apply this technique to perform polynomial division modulo I. In the example above, we could have asked whether $xy+1$ *divides* y^2+1 modulo $\langle x^2+1\rangle$. We would obtain the answer *yes*, along with a quotient $(1-xy)$.

Ideal Quotients and Simplification

The ideal quotient operation is the key to our method. Once again, we use Gröbner bases for the actual computations.

Definition Let I and J be ideals of $k[x_1,\ldots,x_n]$. The *ideal quotient* $I:J$ is the ideal $\{f \in k[x_1,\ldots,x_n] : f\,h \in I \text{ for all } h \in J\}$.

It may not be immediately obvious what this does, so consider this example. Suppose a and b are univariate polynomials over a field and we compute $\langle a\rangle : \langle b\rangle$. What do we get ?

We must get the ideal of polynomials whose product with b is divisible by a. A minimal element will be $\mathrm{lcm}(a,b)/b =$ *a/gcd(a,b)*, which is a numerator for a simplified fraction equivalent to $\frac{a}{b}$.

An initial method is the following:

- Given a/b, construct $\langle a, I\rangle$ and $\langle b, I\rangle$ in $k[x_1,\ldots,x_n]$.
- Compute the ideal quotient $\langle a, I\rangle : \langle b, I\rangle$ using Gröbner bases, and choose a minimal element $c \notin I$ to be a numerator.
- Divide $b\,c$ by a modulo I using Gröbner bases. This will always succeed. Obtain d with $b\,c \equiv a\,d \mod I$.
- Then a/b reduces to c/d modulo I. □

Reduced Canonical Forms

Unfortunately the method above does not always produce a desirable result. Our solution is to construct the *module* of equivalent fractions, building on the quotient construction above. Given a/b, we generate

$$M = \{(c,d) : ad - bc \equiv 0 \mod I\}$$

as a module over $k[x_1,\ldots,x_n]$. Using Gröbner bases for modules and a *term over position* module order, we compute a fraction whose greatest monomial is minimal with respect to a given monomial order. The result is unique if the numerator and denominator of the fraction are reduced modulo I, where I is a prime ideal.

Minimizing the Total Degree

To address the simplification problem, we must construct fractions with minimal total degree. To wit:

- If the polynomials and the ideal are *homogeneous*, then the initial method above will produces a fraction with minimal total degree, provided the chosen numerator has minimal total degree.
- For the general case we conduct a search. Gröbner basis methods can not be applied, since they operate on only one term at a time.

Future Work

- It is often possible to recover factorizations of f modulo I from a prime decomposition of $\sqrt{\langle f, I\rangle}$.
- The idea is promising and it is consistent with our understanding, however major technical obstacles remain.
- We are working to address these issues with the goal of developing a robust algorithm for computing factorizations in $k[x_1,\ldots,x_n]\,/\,I$.

MITACS Symbolic Analysis Project – Supervisor Peter Borwein
Centre for Experimental and Constructive Mathematics
Computer Algebra Group
Simon Fraser University